On Strategic Defense in Stochastic Networks

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Stochastic Network Cumulative Loss Model with Delayed Observation

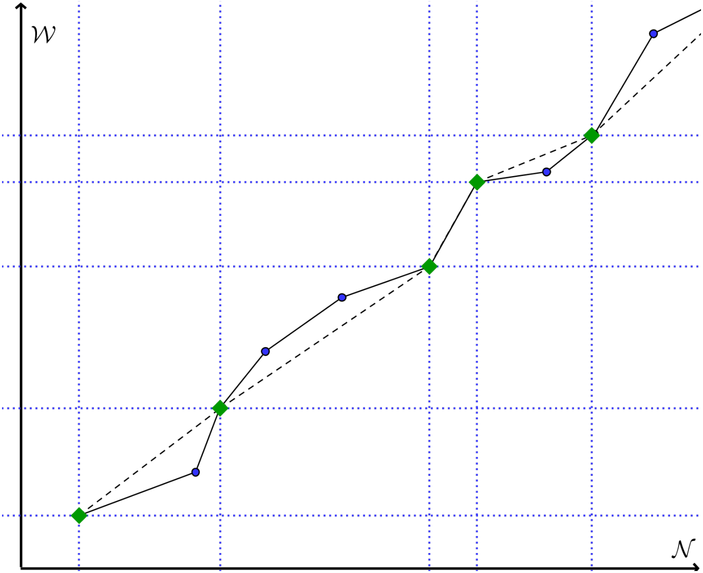
- $\{t_k\}_{k\in\mathbb{N}}$ – point process of attack times
- n_k – *iid* nodes lost at t_k with PGF $g(z) = \mathbb{E}\left[z^{n_1}\right]$
- w_{jk} – *iid* weights with LST $l(u) = \mathbb{E}\left[e^{-uw_{11}}\right]$
  - $w_k = \sum_{j=1}^{n_k} w_{jk}$ – total weight lost at t_k
- Therefore, losses in $[0,t]$ form a monotone increasing process on $\mathbb{N} \times \mathbb{R}_{\geq 0}$,

$$\eta([0,t]) = \sum_{k=1}^{\infty} (n_k, w_k)\varepsilon_{t_k}([0,t]) \qquad \textbf{(Real-time process)}$$

- Process is bserved upon a delayed renewal process $\{\tau_j\}_{j\in\mathbb{Z}_{\geq 0}}$

$$(N_n, W_n) = \sum_{j=0}^{\infty} \eta((\tau_{j-1}, \tau_j])\varepsilon_{\tau_j}([0,\tau_n]) = \sum_{j=0}^{n} \eta((\tau_{j-1}, \tau_j]) \quad \textbf{(Embedded process)}$$

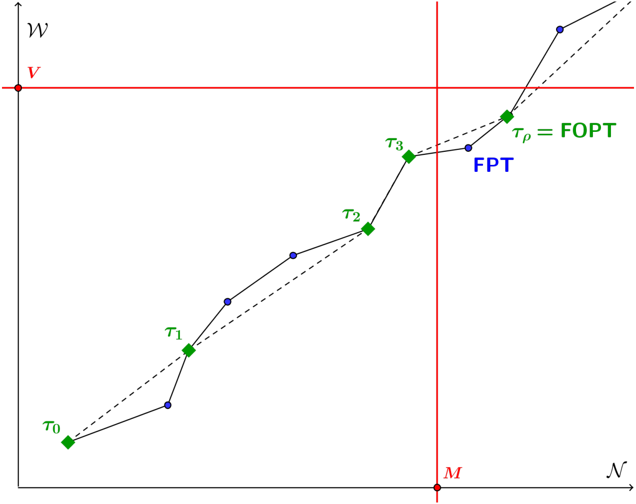
Delayed Observation



Observed Threshold Crossings

- The **first observed passage time** (FOPT) is τ_ρ, where

$$\rho = \min\{n \in \mathbb{Z}_{\geq 0} : (N_n, W_n) \notin [0, M] \times [0, V]\}$$



Functional of Interest

- We seek a joint functional of the of the process upon $\tau_{\rho-1}$ and τ_ρ,

$$\Phi(\alpha_0, \alpha, \beta_0, \beta, h_0, h) = \mathbb{E}\left[{\alpha_0}^{N_{\rho-1}} \alpha^{N_\rho} e^{-\beta_0 W_{\rho-1} - \beta W_\rho} e^{-h_0 \tau_{\rho-1} - h\tau_\rho}\right]$$

- Φ $\to$ marginal transforms $\to$ moments and distributions
  - $\Phi(1, \alpha, 0, 0, 0, 0) = \mathbb{E}\left[\alpha^{N_\rho}\right]$
  - $\Phi(1, 1, 0, 0, h_0, 0) = \mathbb{E}\left[e^{-h_0 \tau_{\rho-1}}\right]$
- The goal is to derive Φ in an analytically or numerically tractable form
- Using an operator $\mathcal{H}$ (a composition of modified Laplace and z-transforms)

$$\Phi \xrightarrow{\mathcal{H}} \Psi \xrightarrow{\text{Assumptions}} \Psi \text{ (convenient form) } \xrightarrow{\mathcal{H}^{-1}} \Phi \text{ (tractable)}$$

Results for a Special Case

1. $\{t_k\}$ form a Poisson point process of rate λ on $\mathbb{R}_{\geq 0}$
2. $\Delta_k \in [\text{Exponential}(\mu)] \implies L(\theta) = \frac{\mu}{\mu+\theta}$
3. $n_k \in [\text{Geometric}(a)] \implies g(z) = \frac{az}{1-bz}, (b = 1-a)$
4. $w_{jk} \in [\text{Exponential}(\xi)] \implies l(u) = \frac{\xi}{\xi+u}$

- Under these assumptions, we find the CDF of τ_ρ
  - $\mathbb{P}(\tau_\rho < \vartheta) = \lambda P(M-1, \xi V) \sum_{j=0}^{M-1} c_j \phi_j(\vartheta) + \lambda e^{-\xi\lambda} \sum_{k=0}^{M-2} \frac{(\xi V)^k}{k!} \sum_{j=0}^{k} d_j \phi_j(\vartheta)$
    - $\phi_j(\vartheta) = \frac{1}{\lambda^{j+1}} P(j+1, \lambda\vartheta) - \frac{e^{-\mu\vartheta}}{(\lambda-\mu)^{j+1}} P(j+1, (\lambda-\mu)\vartheta)$

Useful Results: Means at τ_ρ

$$\mathbb{E}[N_\rho] = \frac{\lambda + b\mu}{a\mu} + M - (M-1)Q(M-1,\xi V) + \xi V Q(M-2,\xi V)$$

$$\mathbb{E}[W_\rho] = \frac{\mathbb{E}[N_\rho]}{\xi} = \mathbb{E}[N_\rho]\,\mathbb{E}[w_{11}]$$

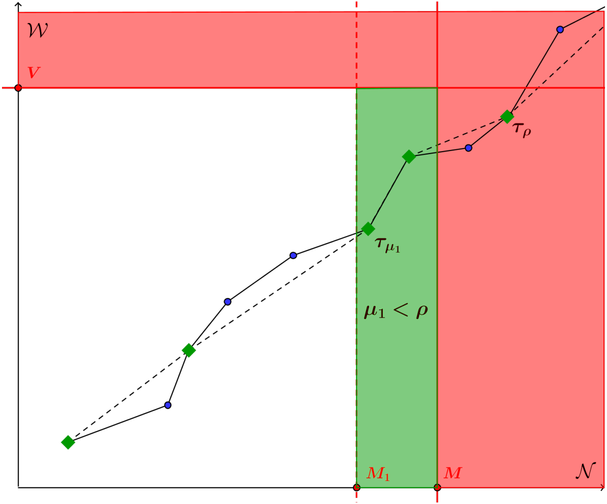
We simulated 1,000 realizations of the process under each of the some sets of parameters $(\lambda, \mu, a, \xi, M, V)$ and recorded the sample means:

Parameters	$\mathbb{E}[N_\rho]$	S. Mean	A. Error	$\mathbb{E}[W_\rho]$	S. Mean	A. Error
(1)	989.08	988.82	0.26	989.08	990.06	0.98
(2)	990.63	990.39	0.24	990.63	990.27	0.36
(3)	989.28	989.92	0.64	989.28	988.97	0.31
(4)	989.08	989.08	0.00	989.08	989.68	0.31
(5)	503.00	502.73	0.27	1006.00	1005.04	0.96
(6)	1002.00	1001.57	0.43	501.00	500.91	0.09
(7)	803.00	802.68	0.32	803.00	802.67	0.33
(8)	752.00	752.10	0.10	752.00	751.68	0.32
(9)	493.57	493.46	0.11	987.14	986.59	0.55

J. H. Dshalalow and R. White. *Neural, Parallel, and Scientific Computations*, 21 (2013).

Auxiliary Threshold Model

We introduce another threshold $M_1 < M$ with $\mu_1 = \min\{n : N_n > M_1\}$, and consider the case where $\mu_1 < \rho$



Functional of Interest

$$\Phi_{\mu_1<\rho} = \Phi_{\mu_1<\rho}\left(z_0, z, \alpha_0, \alpha, v_0, v, \beta_0, \beta, \theta_0, \theta, h_0, h\right)$$
$$= E\left[\boxed{z_0^{N_{\mu_1-1}} z^{N_{\mu_1}}}\alpha_0^{N_{\rho-1}}\alpha^{N_\rho}\boxed{e^{-v_0 W_{\mu_1-1}-vW_{\mu_1}}}e^{-\beta_0 W_{\rho-1}-\beta W_\rho}\right.$$
$$\left.\times\boxed{e^{-\theta_0\tau_{\mu_1-1}-\theta\tau_{\mu_1}}}e^{-h_0\tau_{\rho-1}-h\tau_\rho}\boxed{\mathbf{1}_{\{\mu_1<\rho\}}}\right]$$

New interesting results

$$\Phi_{\mu_1<\rho}\left(1,1,1,1,0,0,0,0,0,-h,0,h\right) = E\left[e^{-h\left(\tau_\rho-\tau_{\mu_1}\right)}\right]$$
$$\Phi_{\mu_1<\rho}(1,1,1,1,0,0,0,0,0,0,0,0) = \mathbb{E}\left[\mathbf{1}_{\{\mu_1<\rho\}}\right] = \mathbb{P}(\mu_1<\rho)$$

With another operator $\mathcal{D}_{\mu_1}$, the path to results remains the same

$$\Phi_{\mu_1<\rho} \xrightarrow{\mathcal{D}_{\mu_1}} \Psi_{\mu_1<\rho} \xrightarrow{\text{Assumptions}} \Psi_{\mu_1<\rho}\ \text{(explicit)} \xrightarrow{\mathcal{D}_{\mu_1}^{-1}} \Phi_{\mu_1<\rho}\ \text{(tractable)}$$

Strategy and Model 1: Constant Observation

In this model, we have

1. $\Delta_k = c \ a.s.$
2. n_k with **arbitrary** finite distribution $(p_1, p_2, ..., p_m)$
3. $w_{jk} \in [\text{Gamma}(\alpha, \xi)]$

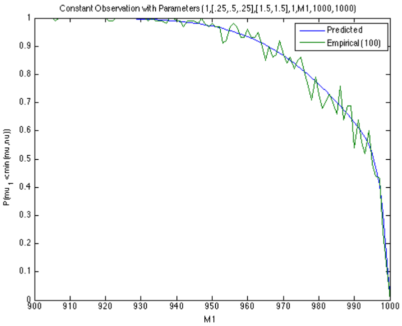
$$\Phi_{\mu_1<\rho}(1, z, 1, 1, 0, v, 0, 0, 0, \theta, 0, 0) = E\left[z^{N_{\mu_1}} e^{-vW_{\mu_1}} e^{-\theta\tau_{\mu_1}} \mathbf{1}_{\{\mu_1<\rho\}}\right]$$

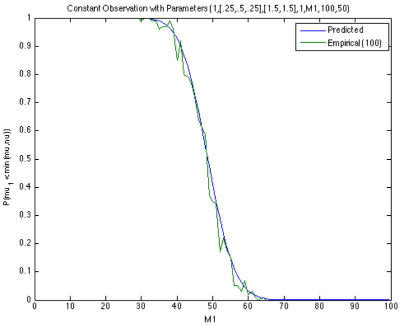
$$= \left\{ \sum_{k=0}^{M_1-1} z^k F_k \sum_{m=0}^{M-1-k} z^m E_m \left(\frac{\xi}{v+\xi}\right)^{\alpha(k+m)} P(\alpha(k+m), (v+\xi)V) \right.$$

$$\left. - \sum_{k=0}^{M_1-1} z^k \left(\frac{\xi}{v+\xi}\right)^{\alpha k} P(\alpha k, (v+\xi)V) \sum_{n=0}^{k} E_n F_{k-n} \right\} e^{-c(\theta+\lambda)}$$

$$F_j = \sum_{r=0}^{\left\lfloor \frac{R-1}{R} j \right\rfloor} (c\lambda)^{j-r} \boxed{Li_{-(j-r)}\left(e^{-c(\theta+\lambda)}\right)} \sum_{\substack{\|\boldsymbol{\beta}\|_1=j \\ [R]\cdot\boldsymbol{\beta}=r+j}} \frac{p_1^{\beta_1} \cdots p_R^{\beta_R}}{\beta_1! \ \cdots \ \beta_R!}$$

We find $\Phi_{\mu_1<\rho}(1,1,1,1,0,0,0,0,0,0,0,0) = E\left[\mathbf{1}_{\{\mu_1<\rho\}}\right] = P(\mu_1 < \rho)$ and compare to simulated results for a range of M_1 values





J. H. Dshalalow and R. White. *Stochastic Analysis and Applications*, 32:3 (2014).

Current Work: Time Sensitive Analysis

$$\Phi_1(t) = \mathbb{E}\left[v^{N_{\rho-1}} u^{N_\rho} \boxed{z^{N(t)}} e^{-aW_{\rho-1} - bW_\rho - \boxed{wW(t)}} e^{-h_0 \tau_{\rho-1} - h\Delta_\rho} \boxed{\mathbf{1}_{\{t<\tau_{\rho-1}\}}}\right]$$

$$\Phi_2(t) = \mathbb{E}\left[v^{N_{\rho-1}} u^{N_\rho} \boxed{z^{N(t)}} e^{-aW_{\rho-1} - bW_\rho - \boxed{wW(t)}} e^{-h_0 \tau_{\rho-1} - h\Delta_\rho} \boxed{\mathbf{1}_{\{\tau_{\rho-1} \le t<\tau_{\rho}\}}}\right]$$

$$\Phi(t) = \mathbb{E}\left[v^{N_{\rho-1}} u^{N_\rho} \boxed{z^{N(t)}} e^{-aW_{\rho-1} - bW_\rho - \boxed{wW(t)}} e^{-h_0 \tau_{\rho-1} - h\Delta_\rho} \boxed{\mathbf{1}_{\{t<\tau_{\rho}\}}}\right]$$

New capabilities:

- Joint results: $\mathbb{E}\left[u^{N_\rho}\mathbf{1}_{\{t<\tau_\rho\}}\right] \implies \mathbb{P}\left\{N_\rho = n, \tau_\rho > t\right\}$
- Conditional probabilities: $\mathbb{P}\{N_\rho = n | \tau_\rho > t\} = \frac{\mathbb{P}\{N_\rho = n, \tau_\rho > t\}}{\mathbb{P}\{\tau_\rho > t\}}$

J. H. Dshalalow and R. White. Time Sensitive Analysis of Independent and Stationary Increments Processes (2015) [Submitted]

Current Work: Extensions

- d-Dimensional Process
  - For $\mathbf{\Delta} = (S_1, S_2 - S_1) = (\Delta_1, \Delta_2)$,

$$\Phi_1(t, \mathbf{v}_1, \mathbf{v}_2, \mathbf{w}, \mathbf{x}) = \mathbb{E}\left[e^{-\mathbf{v}_1 \cdot \mathbf{N}(S_1) - \mathbf{v}_2 \cdot \mathbf{N}(S_2) - \mathbf{w} \cdot \mathbf{N}(t) - \mathbf{x} \cdot \mathbf{\Delta}} \mathbf{1}_{\{t < S_1\}}\right]$$

$$\Phi_2(t, \mathbf{v}_1, \mathbf{v}_2, \mathbf{w}, \mathbf{x}) = \mathbb{E}\left[e^{-\mathbf{v}_1 \cdot \mathbf{N}(S_1) - \mathbf{v}_2 \cdot \mathbf{N}(S_2) - \mathbf{w} \cdot \mathbf{N}(t) - \mathbf{x} \cdot \mathbf{\Delta}} \mathbf{1}_{\{S_1 \le t < S_1 + \Delta_1\}}\right]$$

- m Times of Interest in d Dimensions
  - For $S_0 = 0$, $1 \le n \le m$, and $\mathbf{\Delta} = (\Delta_1, ..., \Delta_m)$,

$$\Phi_n(t, \mathbf{v}_1, ..., \mathbf{v}_m, \mathbf{w}, \mathbf{x}) = \mathbb{E}\left[e^{-\sum_{j=1}^{m} \mathbf{v}_j \cdot \mathbf{N}(S_j) - \mathbf{w} \cdot \mathbf{N}(\mathbf{t}) - \mathbf{x} \cdot \mathbf{\Delta}} \mathbf{1}_{\{S_{n-1} \le t < S_{n-1} + \Delta_n\}}\right]$$

J. H. Dshalalow and R. White. Time Sensitive Analysis of Monotone Multidimensional Random Walks (2015) [In preparation]