Problem 12.5.15

(a) Find the symmetric equations for for the line that passes through the point $(1, -5, 6)$ and is parallel to the vector $\langle -1, 2, -3 \rangle$.

(b) Find the points in which the required line in part (a) intersects the coordinate planes

For part (a), we know $(1, -5, 6)$ is a point on the line, and that the line is parallel to $\langle -1, 2, -3 \rangle$, so we can take this vector itself as our direction vector, so we have

$$\boldsymbol{v} = \langle -1, 2, -3 \rangle, \qquad \boldsymbol{r}_0 = \langle 1, -5, 6 \rangle \tag{1}$$

The symmetric equations of a line are in this form:

$$\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c} \tag{2}$$

for $\boldsymbol{v} = \langle a, b, c \rangle$ and $\boldsymbol{r}_0 = (x_0, y_0, z_0)$, so we can just plug in the values from (1) above to find the symmetric equations for our line:

$$\frac{x - 1}{-1} = \frac{y + 5}{2} = \frac{z - 6}{-3} \tag{3}$$

For part (b), we want to know where the line intersects the coordinate planes – that is, the points where $x = 0$, $y = 0$, or $z = 0$, so we just plug in each one to find the points.

First, let's plug in $x = 0$ and solve for y and z to find the point of intersection of the yz-plane. Solving for y:

$$\begin{aligned} \frac{0 - 1}{-1} &= \frac{y + 5}{2} \\ 1 &= \frac{y + 5}{2} \\ 2 &= y + 5 \\ -3 &= y \end{aligned}$$

Solving for z:

$$\begin{aligned} \frac{0 - 1}{-1} &= \frac{z - 6}{-3} \\ 1 &= \frac{z - 6}{-3} \\ -3 &= z - 6 \\ 3 &= z \end{aligned}$$

Therefore, the intersection with the yz-plane is $(0, -3, 3)$

We did this in class, as well as found by plugging in $y = 0$ and solving for x and z, the xz-intersect is $\left(-\frac{3}{2}, 0, -\frac{3}{2}\right)$.

After this, I claimed we were done because we can only have 2 intercepts, but this isn't true – the drawing of the coordinate plane only showed that the line could have at most two coordinate plane intercepts *on the positive side of the axes.*

Rather, a line must hit all of the coordinate planes (maybe on the negative side) unless it is parallel to one of them, so the last part is to plug in $z = 0$ and solve for x and y.

$$\begin{aligned} \frac{0-6}{-3} &= \frac{x-1}{-1} \\ 2 &= \frac{x-1}{-1} \\ -2 &= x-1 \\ -1 &= x \end{aligned}$$

and solve for y:

$$\begin{aligned} \frac{0-6}{-3} &= \frac{y+6}{2} \\ 2 &= \frac{y+5}{2} \\ 4 &= y+5 \\ -1 &= y \end{aligned}$$

So our xy-intercept is $(-1, -1, 0)$.

Problem 12.5.35

Find an equation of the plane that passes through the point $(6, 0, -2)$ and contains the line $x = 4 - 2t$, $y = 3 + 5t$, and $z = 7 + 4t$.

In class, we did get to the correct answer here, but not in a smooth fashion, so let's start from scratch. To write an equation of a plane, we need two things:

1. A point on the plane, which is given here as $(6, 0, -2)$
2. A normal vector from the plane (that is, a vector orthogonal to the plane), typically denoted $\boldsymbol{n}$

Since the given line is contained entirely on the plane, and its direction vector is $\boldsymbol{v}_1 = \langle -2, 5, 4 \rangle$, we know this vector lies on the plane – that is, from any point on the plane, we can go infinitely far in the direction of this vector and remain on the plane.

If we can find another vector lying on the plane $\boldsymbol{v}_2$, we would know that the cross product $\boldsymbol{v}_1 \times \boldsymbol{v}_2$ is orthogonal to both $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$, and will therefore be orthogonal to the plane, which will be our normal vector $\boldsymbol{n}$.

How can we find this $\boldsymbol{v}_2$? We have the point $(6, 0, -2)$ as well as the given line on the plane, so if we pick out one point on the line, we will know the vector between $(6, 0, -2)$ and this point lies on the plane. So plug in $t = 0$ to find that the point $(4, 3, 7)$ is on the line (and therefore on the plane), so the vector from $(6, 0, -2)$ to $(4, 3, 7)$ will be our $\boldsymbol{v}_2$, which we can find by subtracting them from one another (the order doesn't matter):

$$\boldsymbol{v}_2 = \langle 6-4, 0-3, -2-7 \rangle = \langle 2, -3, -9 \rangle \tag{4}$$

Next, we can find our normal vector:

$$\begin{aligned} \boldsymbol{n} = \boldsymbol{v}_1 \times \boldsymbol{v}_2 &= \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ -2 & 5 & 4 \\ 2 & -3 & -9 \end{vmatrix} \\ &= \boldsymbol{i}[(5)(-9) - (4)(-3)] - \boldsymbol{j}[(-2)(-9) - (4)(2)] + \boldsymbol{k}[(-2)(-3) - (5)(2)] \\ &= -33\boldsymbol{i} - 10\boldsymbol{j} - 4\boldsymbol{k} \end{aligned}$$

This is our normal vector, and we have the point on the plane, so we can write the linear equation of the plane as follows:

$$-33x - 10y - 4z = d \tag{5}$$

To solve for d, we can just plug in the point on the plane we know:

$$\begin{aligned} -33(6) - 10(0) - 4(-2) &= d \\ -198 + 8 &= d \\ -190 &= d \end{aligned}$$

Plugging this d value into formula (5) above, we get the equation of the desired plane:

$$-33x - 10y - 4z = -190 \tag{6}$$

The solution in your book has the same equation, but with each side multiplied by -1. With any equation, we can multiply both sides by the same number and get another valid form of the equation, so this one is equally valid:

$$33x + 10y + 4z = 190 \tag{7}$$

Problem 12.5.33

Find the equation of the plane passing through points $P(3,-1,2)$, $Q(8,2,4)$, and $R(-1,-2,-3)$.

We didn't try this problem in class, but I had hoped to do a similar problem. Similar to the previous problem, we have points on the plane already, and we need to find a normal vector.

As with the previous problem, we will seek to find two vectors lying on the plane and then find their cross product to be the normal vector. Since P, Q, and R are all on the plane, the vector connecting the points must also lie on the plane, so we can use two of these, say $\vec{PQ}$ and $\vec{PR}$.

$$\vec{PQ} = Q - P = \langle 8-3, 2-(-1), 4-2\rangle = \langle 5,3,2\rangle \tag{8}$$

$$\vec{PR} = R - P = \langle -1-3, -2-(-1), -3-2\rangle = \langle -4,-1,-5\rangle \tag{9}$$

To find the normal vector, we just find the cross product of these two since they both lie on the plane:

$$\begin{aligned} \boldsymbol{n} = \vec{PQ} \times \vec{PR} &= \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ 5 & 3 & 2 \\ -4 & -1 & -5 \end{vmatrix} \\ &= \boldsymbol{i}[-15-(-2)] - \boldsymbol{j}[-25-(-8)] + \boldsymbol{k}[-5-(-12)] \\ &= -13\boldsymbol{i} + 17\boldsymbol{j} + 7\boldsymbol{k} \end{aligned}$$

Therefore, we know our plane equation will be of the form

$$-13x + 17y + 7z = d \tag{10}$$

Plugging in a point we know is on the plane $P = (3,-1,2)$, we solve for d:

$$\begin{aligned} -13(3) + 17(-1) + 7(2) &= d \\ -39 - 17 + 14 &= d \\ -42 &= d \end{aligned}$$

Then the linear equation of the plane is

$$-13x + 17y + 7z = -42 \tag{11}$$