Exiting Patterns of Multivariate Renewal-Reward Processes

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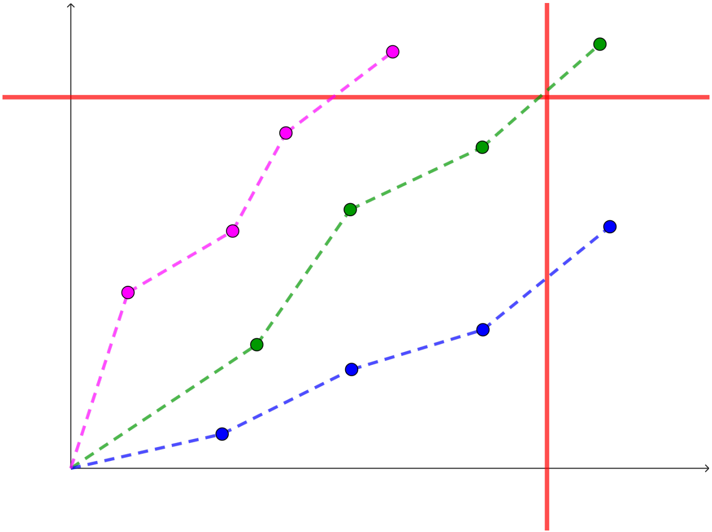
An Exiting Problem in 2D



Nonnegative Random Vectors

- Probability space $(\Omega, \mathcal{F}, \mathbb{P})$
- Consider a sequence of *i.i.d.* nonnegative random vectors

$$X^{[j]} : (\Omega, \mathcal{F}) \to (\mathbb{R}^d_{\geq 0}, \mathcal{B})$$

- Components may be **dependent**, discrete, or continuous

$$X^{[j]} = (X_1^{[j]}, ..., X_d^{[j]})$$

- Each random vector has a common Laplace-Stieltjes transform

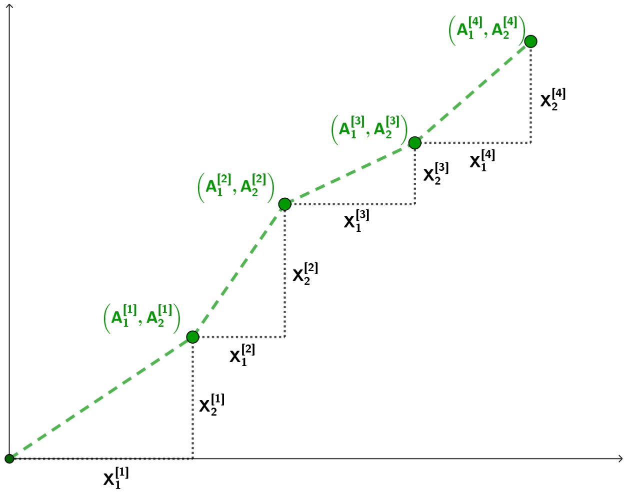
$$\gamma(x) = \mathbb{E}\left[e^{x \cdot X^{[j]}}\right]$$

A Stochastic Process

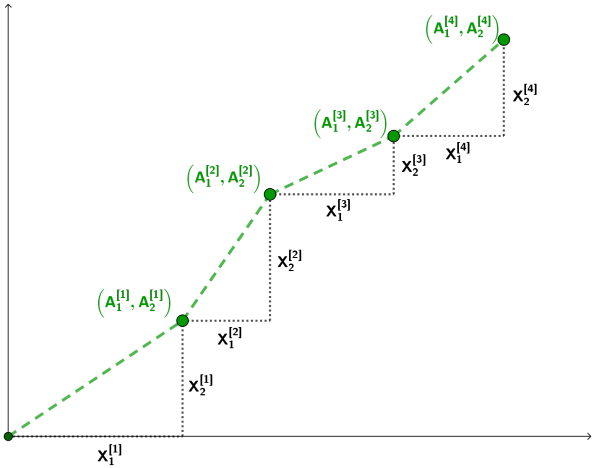
- We study a discrete-time stochastic process summing these random vectors

$$
\begin{aligned}
A^{[1]} &= X^{[1]} \\
A^{[2]} &= X^{[1]} + X^{[2]} \\
A^{[3]} &= X^{[1]} + X^{[2]} + X^{[3]} \\
&\vdots \\
A^{[n]} &= \sum_{j=1}^{n} X^{[j]}
\end{aligned}
$$

- Geometrically, $X^{[j]}$ is the jth "jump" of the process

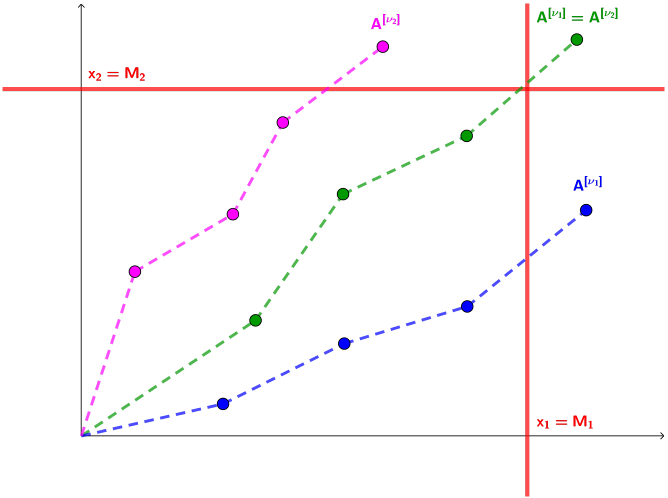

$\left(A_1^{[4]}, A_2^{[4]}\right)$
$X_2^{[4]}$
$\left(A_1^{[3]}, A_2^{[3]}\right)$
$X_1^{[4]}$
$X_2^{[3]}$
$\left(A_1^{[2]}, A_2^{[2]}\right)$
$X_1^{[3]}$
$X_2^{[2]}$
$\left(A_1^{[1]}, A_2^{[1]}\right)$
$X_1^{[2]}$
$X_2^{[1]}$
$X_1^{[1]}$

Applications of Such Processes



- **Stochastic networks**
- Insurance
- **Queueing theory**
- Finance
- Stochastic games
- Reliability theory
- Anomalous diffusion
- **Intrusion detection**

"Exits" of the Process



- **Thresholds** in each dimension

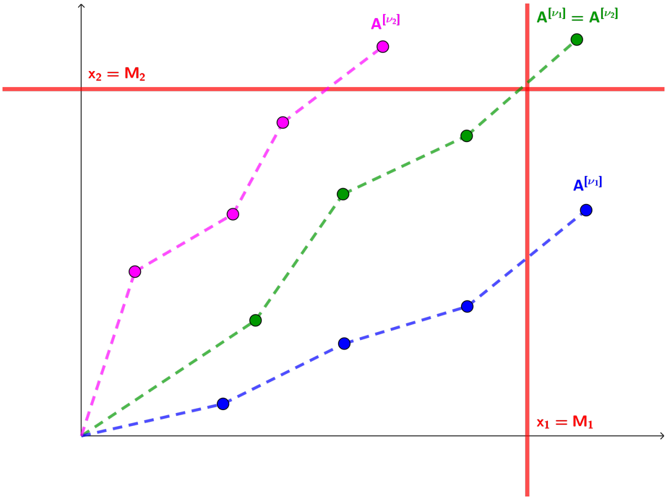
$$M_1,\ M_2,\ \dots,\ M_d$$

- **Exit indices** in each dimension

$$\nu_k = \inf\left\{n : A_k^{[n]} > M_k\right\}$$

- All thresholds will eventually be crossed with probability 1

$$\mathbb{P}(\nu_k < \infty) = 1$$



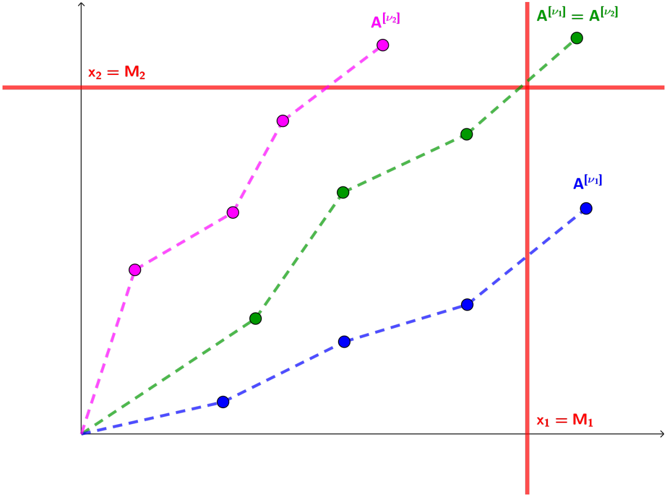
Question: In what **order** will the thresholds be crossed?

$\nu_1 < \nu_2$?

$\nu_1 = \nu_2$?

$\nu_1 > \nu_2$?

"Exits" of the Process



Each weak ordering could happen, so we seek **probabilities**

$$\mathbb{P}(\nu_1 < \nu_2)$$

$$\mathbb{P}(\nu_1 = \nu_2)$$

$$\mathbb{P}(\nu_1 > \nu_2)$$

Prior Study

- For $d = 2$, these ideas are implicit in Dshalalow and White [2013] in the context of reliability analysis of stochastic networks, but probabilities were not addressed

- For $d = 2$, these ideas were explicitly studied in White [2015] and further used in stochastic networks

- For $d = 3$, similar ideas are implicit in Dshalalow [1997], although the aim was finding results about on exit times and locations of renewal processes, but probabilities were not addressed

- For $d = 4$, similar ideas are implicit in Dshalalow and Liew [2006c] in the context of random walk processes applied to finance, but probabilities were not addressed

Deriving $\mathbb{P}(\nu_1 < \nu_2)$ in 2D

- $\{\nu_1 < \nu_2\}$ can be partitioned into disjoint sub-events $\{\nu_1 = j_1, \nu_2 = j_2\}$ for all $j_1, j_2 \in \mathbb{N}$ with $j_2 > j_1$

$$\{\nu_1 < \nu_2\} = \bigcup_{j_1=1}^{\infty} \bigcup_{j_2=j_1+1}^{\infty} \{\nu_1 = j_1, \nu_2 = j_2\}$$

- To compute the probability, we can compute

$$\mathbb{P}(\nu_1 < \nu_2) = \sum_{j_1=1}^{\infty} \sum_{j_2=j_1+1}^{\infty} \mathbb{P}(\nu_1 = j_1, \nu_2 = j_2)$$

$$\mathbb{E}\left[\mathbb{1}_{\{\nu_1 < \nu_2\}}\right] = \sum_{j_1=1}^{\infty} \sum_{j_2=j_1+1}^{\infty} \mathbb{E}\left[\mathbb{1}_{\{\nu_1 = j_1,\, \nu_2 = j_2\}}\right]$$

Deriving $\mathbb{P}(\nu_1 < \nu_2)$ in 2D - Strategy

- We want $\mathbb{P}(\nu_1 < \nu_2)$ in an analytically or numerically **tractable form**
- **Operational calculus** strategy to derive $\mathbb{P}(\nu_1 < \nu_2)$

$$\mathbb{P}(\nu_1 < \nu_2) \xrightarrow{\mathcal{H}} \mathbb{Q}(\nu_1 < \nu_2) \xrightarrow{\text{Assumptions}} \mathbb{Q}(\nu_1 < \nu_2) \text{ (convenient form) } \xrightarrow{\mathcal{H}^{-1}} \mathbb{P}(\nu_1 < \nu_2) \text{ (tractable)}$$

Operators in 2D

- Let $q = (q_1, q_2) \in \mathbb{R}^2_{\geq 0}$ and consider the operator

$$\mathcal{H}_q(\cdot)(x) = \mathcal{H}_{q_1} \circ \mathcal{H}_{q_2}(\cdot)(x_1, x_2)$$

- The operator $\mathcal{H}_{q_k}$ take one of two forms

$$\mathcal{H}_{q_k}(\cdot) = \begin{cases} \mathcal{LC}_{q_k}(\cdot)(x_k), & \text{if } X_k^{[j]} \text{ are continuous in } \mathbb{R}_{\geq 0} \\ D_{q_k}(\cdot)(x_k), & \text{if } X_k^{[j]} \text{ are discrete in } \mathbb{N} \end{cases}$$

where

$$\mathcal{LC}_{q_k}(f(q_k))(x_k) = x_k \int_{q_k=0}^{\infty} e^{-x_k q_k} f(q_k)\, dq_k \text{ for } x_k > 0$$

and

$$D_{q_k}(f(q_k))(x_k) = (1 - x_k) \sum_{q_k=0}^{\infty} x_k^{q_k} f(q_k) \text{ for } \|x_k\| < 1$$

Lemma 1

- Let $\nu_k(q) = \inf\left\{n : A_k^{[n]} > q\right\}$
- Continuous components

$$\mathcal{LC}_q(\mathbb{1}_{\{\nu_k(q)=j\}})(x) = x\int_0^\infty e^{-xq}\mathbb{1}_{\{\nu_k(q)=j\}}\,dq = x\int_{A_k^{[j-1]}}^{A_k^{[j]}} e^{-xq}\,dq = e^{-xA_k^{[j-1]}} - e^{-xA_k^{[j]}}$$

- Discrete components (from Dshalalow [2013])

$$D_q\left(\mathbb{1}_{\{\nu_k(q)=j\}}\right)(x) = (1-x)\sum_{q=0}^{\infty} x^q\mathbb{1}_{\{\nu_k(q)=j\}} = (1-x)\sum_{q=A_k^{[j-1]}}^{A_k^{[j]}-1} x^q = x^{A_k^{[j-1]}} - x^{A_k^{[j]}}.$$

Deriving $\mathbb{P}(\nu_1 < \nu_2)$ in 2D - Series Expansion

- Let $W(q) = \{\nu_1(q_1) < \nu_2(q_2)\}$, then

$$\mathbb{E}\left[\mathbb{1}_{W(q)}\right] = \sum_{j_1=1}^{\infty} \sum_{j_2=j_1+1}^{\infty} \mathbb{E}\left[\mathbb{1}_{\{\nu_1(q_1)=j_1,\, \nu_2(q_2)=j_2\}}\right]$$

$$\mathcal{H}_q\left[\mathbb{E}\left[\mathbb{1}_{W(q)}\right]\right](x) = \sum_{j_1=1}^{\infty} \sum_{j_2=j_1+1}^{\infty} \mathbb{E}\left[\mathcal{H}_q\left[\mathbb{1}_{\{\nu_1(q_1)=j_1,\, \nu_2(q_2)=j_2\}}\right](x)\right] \qquad \text{(Fubini's Theorem)}$$

$$\mathcal{H}_q\left[\mathbb{P}(W(q))\right](x) = \sum_{j_1=1}^{\infty} \sum_{j_2=j_1+1}^{\infty} \mathbb{E}\left[\left(x_1^{A_1^{[j_1-1]}} - x_1^{A_1^{[j_1]}}\right)\left(x_2^{A_2^{[j_2-1]}} - x_2^{A_2^{[j_2]}}\right)\right] \qquad \text{(Lemma 1)}$$

Deriving $\mathbb{P}(\nu_1 < \nu_2)$ in 2D - Series Expansion

- Let $W(q) = \{\nu_1(q_1) < \nu_2(q_2)\}$, then

$$\mathbb{E}\left[\mathbb{1}_{W(q)}\right] = \sum_{j_1=1}^{\infty} \sum_{j_2=j_1+1}^{\infty} \mathbb{E}\left[\mathbb{1}_{\{\nu_1(q_1)=j_1,\, \nu_2(q_2)=j_2\}}\right]$$

$$\mathcal{H}_q\left[\mathbb{E}\left[\mathbb{1}_{W(q)}\right]\right](x) = \sum_{j_1=1}^{\infty} \sum_{j_2=j_1+1}^{\infty} \mathbb{E}\left[\mathcal{H}_q\left[\mathbb{1}_{\{\nu_1(q_1)=j_1,\, \nu_2(q_2)=j_2\}}\right](x)\right] \qquad \text{(Fubini's Theorem)}$$

$$\mathcal{H}_q\left[\mathbb{P}(W(q))\right](x) = \sum_{j_1=1}^{\infty} \sum_{j_2=j_1+1}^{\infty} \mathbb{E}\left[\left(x_1^{A_1^{[j_1-1]}} - x_1^{A_1^{[j_1]}}\right)\left(x_2^{A_2^{[j_2-1]}} - x_2^{A_2^{[j_2]}}\right)\right] \qquad \text{(Lemma 1)}$$

Deriving $\mathbb{P}(\nu_1 < \nu_2)$ in 2D - Expanding

$$\left(x_1^{A_1^{[j_1-1]}} - x_1^{A_1^{[j_1]}}\right)\left(x_2^{A_2^{[j_2-1]}} - x_2^{A_2^{[j_2]}}\right)$$

$$x_1^{A_1^{[j_1-1]}}\left(1 - x_1^{X_1^{[j_1]}}\right) x_2^{A_2^{[j_2-1]}}\left(1 - x_2^{X_2^{[j_2]}}\right)$$

$$x_1^{A_1^{[j_1-1]}}\left(1 - x_1^{X_1^{[j_1]}}\right) x_2^{A_2^{[j_1-1]}+X_2^{[j_1]}+X_2^{[j_1+1]}+\cdots+X_2^{[j_2-1]}}\left(1 - x_2^{X_2^{[j_2]}}\right)$$

Deriving $\mathbb{P}(\nu_1 < \nu_2)$ in 2D - 4 Terms

$$
\begin{aligned}
+ \quad & \left(x_1^{A_1^{[j_1-1]}} x_2^{A_2^{[j_1-1]}}\right) \left(x_2^{X_2^{[j_1]}}\right) \left(x_2^{X_2^{[j_1+1]}+\cdots+X_2^{[j_2-1]}}\right) \\
- \quad & \left(x_1^{A_1^{[j_1-1]}} x_2^{A_2^{[j_1-1]}}\right) \left(x_1^{X_1^{[j_1]}} x_2^{X_2^{[j_1]}}\right) \left(x_2^{X_2^{[j_1+1]}+\cdots+X_2^{[j_2-1]}}\right) \\
- \quad & \left(x_1^{A_1^{[j_1-1]}} x_2^{A_2^{[j_1-1]}}\right) \left(x_2^{X_2^{[j_1]}}\right) \left(x_2^{X_2^{[j_1+1]}+\cdots+X_2^{[j_2-1]}}\right) \left(x_2^{X_2^{[j_2]}}\right) \\
+ \quad & \left(x_1^{A_1^{[j_1-1]}} x_2^{A_2^{[j_1-1]}}\right) \left(x_1^{X_1^{[j_1]}} x_2^{X_2^{[j_1]}}\right) \left(x_2^{X_2^{[j_1+1]}+\cdots+X_2^{[j_2-1]}}\right) \left(x_2^{X_2^{[j_2]}}\right)
\end{aligned}
$$

Deriving $\mathbb{P}(\nu_1 < \nu_2)$ in 2D - Separating by Time

$$+ \left(x_1^{A_1^{[j_1-1]}} x_2^{A_2^{[j_1-1]}}\right)\left(x_2^{X_2^{[j_1]}}\right)\left(x_2^{X_2^{[j_1+1]}+\cdots+X_2^{[j_2-1]}}\right)$$

$$- \left(x_1^{A_1^{[j_1-1]}} x_2^{A_2^{[j_1-1]}}\right)\left(x_1^{X_1^{[j_1]}} x_2^{X_2^{[j_1]}}\right)\left(x_2^{X_2^{[j_1+1]}+\cdots+X_2^{[j_2-1]}}\right)$$

$$- \left(x_1^{A_1^{[j_1-1]}} x_2^{A_2^{[j_1-1]}}\right)\left(x_2^{X_2^{[j_1]}}\right)\left(x_2^{X_2^{[j_1+1]}+\cdots+X_2^{[j_2-1]}}\right)\left(x_2^{X_2^{[j_2]}}\right)$$

$$+ \left(x_1^{A_1^{[j_1-1]}} x_2^{A_2^{[j_1-1]}}\right)\left(x_1^{X_1^{[j_1]}} x_2^{X_2^{[j_1]}}\right)\left(x_2^{X_2^{[j_1+1]}+\cdots+X_2^{[j_2-1]}}\right)\left(x_2^{X_2^{[j_2]}}\right)$$

Deriving $\mathbb{P}(\nu_1 < \nu_2)$ in 2D - Expected Value

$$+ \quad \mathbb{E}\left[\left(x_1^{A_1^{[j_1-1]}} \; x_2^{A_2^{[j_1-1]}}\right)\left(x_2^{X_2^{[j_1]}}\right)\left(x_2^{X_2^{[j_1+1]}+\cdots+X_2^{[j_2-1]}}\right)\right]$$

$$- \quad \mathbb{E}\left[\left(x_1^{A_1^{[j_1-1]}} \; x_2^{A_2^{[j_1-1]}}\right)\left(x_1^{X_1^{[j_1]}} \; x_2^{X_2^{[j_1]}}\right)\left(x_2^{X_2^{[j_1+1]}+\cdots+X_2^{[j_2-1]}}\right)\right]$$

$$- \quad \mathbb{E}\left[\left(x_1^{A_1^{[j_1-1]}} \; x_2^{A_2^{[j_1-1]}}\right)\left(x_2^{X_2^{[j_1]}}\right)\left(x_2^{X_2^{[j_1+1]}+\cdots+X_2^{[j_2-1]}}\right)\left(x_2^{X_2^{[j_2]}}\right)\right]$$

$$+ \quad \mathbb{E}\left[\left(x_1^{A_1^{[j_1-1]}} \; x_2^{A_2^{[j_1-1]}}\right)\left(x_1^{X_1^{[j_1]}} \; x_2^{X_2^{[j_1]}}\right)\left(x_2^{X_2^{[j_1+1]}+\cdots+X_2^{[j_2-1]}}\right)\left(x_2^{X_2^{[j_2]}}\right)\right]$$

Deriving $\mathbb{P}(\nu_1 < \nu_2)$ in 2D - Independent Jumps

$$+ \quad \mathbb{E}\left[x_1^{A_1^{[j_1-1]}} x_2^{A_2^{[j_1-1]}}\right] \mathbb{E}\left[x_2^{X_2^{[j_1]}}\right] \mathbb{E}\left[x_2^{X_2^{[j_1+1]}+\cdots+X_2^{[j_2-1]}}\right]$$

$$- \quad \mathbb{E}\left[x_1^{A_1^{[j_1-1]}} x_2^{A_2^{[j_1-1]}}\right] \mathbb{E}\left[x_1^{X_1^{[j_1]}} x_2^{X_2^{[j_1]}}\right] \mathbb{E}\left[x_2^{X_2^{[j_1+1]}+\cdots+X_2^{[j_2-1]}}\right]$$

$$- \quad \mathbb{E}\left[x_1^{A_1^{[j_1-1]}} x_2^{A_2^{[j_1-1]}}\right] \mathbb{E}\left[x_2^{X_2^{[j_1]}}\right] \mathbb{E}\left[x_2^{X_2^{[j_1+1]}+\cdots+X_2^{[j_2-1]}}\right] \mathbb{E}\left[x_2^{X_2^{[j_2]}}\right]$$

$$+ \quad \mathbb{E}\left[x_1^{A_1^{[j_1-1]}} x_2^{A_2^{[j_1-1]}}\right] \mathbb{E}\left[x_1^{X_1^{[j_1]}} x_2^{X_2^{[j_1]}}\right] \mathbb{E}\left[x_2^{X_2^{[j_1+1]}+\cdots+X_2^{[j_2-1]}}\right] \mathbb{E}\left[x_2^{X_2^{[j_2]}}\right]$$

Deriving $\mathbb{P}(\nu_1 < \nu_2)$ in 2D - Identically Distributed Jumps

$$
\begin{aligned}
&+ \quad \bar{\gamma}(x_1,x_2)^{j_1-1} \quad \bar{\gamma}(1,x_2)^{1} \quad \bar{\gamma}(1,x_2)^{j_2-1-j_1} \quad 1 \\
&- \quad \bar{\gamma}(x_1,x_2)^{j_1-1} \quad \bar{\gamma}(x_1,x_2)^{1} \quad \bar{\gamma}(1,x_2)^{j_2-1-j_1} \quad 1 \\
&- \quad \bar{\gamma}(x_1,x_2)^{j_1-1} \quad \bar{\gamma}(1,x_2)^{1} \quad \bar{\gamma}(1,x_2)^{j_2-1-j_1} \quad \bar{\gamma}(1,x_2)^{1} \\
&+ \quad \bar{\gamma}(x_1,x_2)^{j_1-1} \quad \bar{\gamma}(x_1,x_2)^{1} \quad \bar{\gamma}(1,x_2)^{j_2-1-j_1} \quad \bar{\gamma}(1,x_2)^{1}
\end{aligned}
$$

where $\bar{\gamma}(x_1,x_2) = \gamma(-\ln x_1, -\ln x_2) = \mathbb{E}\left[e^{-\ln x_1 X_1^{[j]} - \ln x_2 X_2^{[j]}}\right] = \mathbb{E}\left[x_1^{X_1^{[j]}} x_2^{X_2^{[j]}}\right]$

Deriving $\mathbb{P}(\nu_1 < \nu_2)$ in 2D - Grouping

$$+ \quad \bar{\gamma}(x_1, x_2)^{j_1-1} \quad \bar{\gamma}(1, x_2)^{1} \quad \bar{\gamma}(1, x_2)^{j_2-1-j_1} \quad 1 \tag{1}$$

$$- \quad \bar{\gamma}(x_1, x_2)^{j_1-1} \quad \bar{\gamma}(x_1, x_2)^{1} \quad \bar{\gamma}(1, x_2)^{j_2-1-j_1} \quad 1 \tag{2}$$

$$- \quad \bar{\gamma}(x_1, x_2)^{j_1-1} \quad \bar{\gamma}(1, x_2)^{1} \quad \bar{\gamma}(1, x_2)^{j_2-1-j_1} \quad \bar{\gamma}(1, x_2)^{1} \tag{3}$$

$$+ \quad \bar{\gamma}(x_1, x_2)^{j_1-1} \quad \bar{\gamma}(x_1, x_2)^{1} \quad \bar{\gamma}(1, x_2)^{j_2-1-j_1} \quad \bar{\gamma}(1, x_2)^{1} \tag{4}$$

$$(1) + (3) = \bar{\gamma}(x_1, x_2)^{j_1-1} \bar{\gamma}(1, x_2) \bar{\gamma}(1, x_2)^{j_2-1-j_1} \left(1 - \bar{\gamma}(1, x_2)\right)$$

$$(2) + (4) = -\bar{\gamma}(x_1, x_2)^{j_1-1} \bar{\gamma}(x_1, x_2) \bar{\gamma}(1, x_2)^{j_2-1-j_1} \left(1 - \bar{\gamma}(1, x_2)\right)$$

$$(1) + (2) + (3) + (4) = \bar{\gamma}(x_1, x_2)^{j_1-1} \bar{\gamma}(1, x_2)^{j_2-1-j_1} \left(1 - \bar{\gamma}(1, x_2)\right) \left(\bar{\gamma}(1, x_2) - \bar{\gamma}(x_1, x_2)\right)$$

Deriving $\mathbb{P}(\nu_1 < \nu_2)$ in 2D - Back to the Series

$$\mathcal{H}_q\left[\mathbb{P}(W(q))\right](x) = \sum_{j_1=1}^{\infty}\sum_{j_2=j_1+1}^{\infty}\mathbb{E}\left[\left(x_1^{A_1^{[j_1-1]}} - x_1^{A_1^{[j_1]}}\right)\left(x_2^{A_2^{[j_2-1]}} - x_2^{A_2^{[j_2]}}\right)\right]$$

$$= \sum_{j_1=1}^{\infty}\sum_{j_2=j_1+1}^{\infty}\bar{\gamma}(x_1,x_2)^{j_1-1}\bar{\gamma}(1,x_2)^{j_2-1-j_1}\left(1-\bar{\gamma}(1,x_2)\right)\left(\bar{\gamma}(1,x_2)-\bar{\gamma}(x_1,x_2)\right)$$

$$= \left(1-\bar{\gamma}(1,x_2)\right)\left(\bar{\gamma}(1,x_2)-\bar{\gamma}(x_1,x_2)\right)\sum_{j_1=1}^{\infty}\bar{\gamma}(x_1,x_2)^{j_1-1}\sum_{j_2=j_1+1}^{\infty}\bar{\gamma}(1,x_2)^{j_2-1-j_1}$$

$$= \left(1-\bar{\gamma}(1,x_2)\right)\left(\bar{\gamma}(1,x_2)-\bar{\gamma}(x_1,x_2)\right)\sum_{j_1=0}^{\infty}\bar{\gamma}(x_1,x_2)^{j_1}\sum_{j_2=0}^{\infty}\bar{\gamma}(1,x_2)^{j_2}$$

Deriving $\mathbb{P}(\nu_1 < \nu_2)$ in 2D - Geometric Series

Lemma (White [2015])

If at least one component of x is inside the unit ball, then $\|\bar{\gamma}(x)\| < 1$.

$$\mathcal{H}_q\left[\mathbb{P}(W(q))\right](x) = \left(1-\bar{\gamma}(1,x_2)\right)\left(\bar{\gamma}(1,x_2)-\bar{\gamma}(x_1,x_2)\right)\sum_{j_1=0}^{\infty}\bar{\gamma}(x_1,x_2)^{j_1}\sum_{j_2=0}^{\infty}\bar{\gamma}(1,x_2)^{j_2}$$

$$= \left(1-\bar{\gamma}(1,x_2)\right)\left(\bar{\gamma}(1,x_2)-\bar{\gamma}(x_1,x_2)\right)\frac{1}{1-\bar{\gamma}(x_1,x_2)}\frac{1}{1-\bar{\gamma}(1,x_2)}$$

$$= \frac{\bar{\gamma}(1,x_2)-\bar{\gamma}(x_1,x_2)}{1-\bar{\gamma}(x_1,x_2)}$$

Deriving $\mathbb{P}(\nu_1 < \nu_2)$ in 2D - Inverting the Operator

$$\mathcal{H}_q\left[\mathbb{P}(W(q))\right](x) = \frac{\bar{\gamma}(1, x_2) - \bar{\gamma}(x_1, x_2)}{1 - \bar{\gamma}(x_1, x_2)}$$

$$\mathbb{P}(W) = \mathcal{H}_x^{-1}\left[\frac{\bar{\gamma}(1, x_2) - \bar{\gamma}(x_1, x_2)}{1 - \bar{\gamma}(x_1, x_2)}\right](M_1, M_2)$$

$$\mathbb{P}(\nu_1 < \nu_2) = \mathcal{H}_x^{-1}\left[\frac{\bar{\gamma}(1, x_2) - \bar{\gamma}(x_1, x_2)}{1 - \bar{\gamma}(x_1, x_2)}\right](M_1, M_2)$$

Given the distribution of $X^{[j]}$, the inverse can be evaluated **explicitly** or **numerically**

Deriving $\mathbb{P}(\nu_1 > \nu_2)$ and $\mathbb{P}(\nu_1 = \nu_2)$ in 2D

$$\mathbb{P}(\nu_1 < \nu_2) = \mathcal{H}_x^{-1}\left[\frac{\bar{\gamma}(1, x_2) - \bar{\gamma}(x_1, x_2)}{1 - \bar{\gamma}(x_1, x_2)}\right](M_1, M_2)$$

$$\mathbb{P}(\nu_1 > \nu_2) = \mathcal{H}_x^{-1}\left[\frac{\bar{\gamma}(x_1, 1) - \bar{\gamma}(x_1, x_2)}{1 - \bar{\gamma}(x_1, x_2)}\right](M_1, M_2)$$

For $\nu_1 = \nu_2$, the problem is easier

$$\mathbb{P}(\nu_1 = \nu_2) = \mathcal{H}_x^{-1}\left[\mathcal{H}_q\left[\sum_{j=0}^{\infty}\mathbb{E}\left[\mathbb{1}_{\{\nu_1(q_1)=\nu_2(q_2)=j\}}\right]\right](x)\right](M_1, M_2)$$

$$= \mathcal{H}_x^{-1}\left[\frac{1 - \bar{\gamma}(x_1, 1) - \bar{\gamma}(1, x_2) + \bar{\gamma}(x_1, x_2)}{1 - \bar{\gamma}(x_1, x_2)}\right](M_1, M_2)$$

Challenges in d Dimensions

- As d grows, the number of weak orderings grows **super-factorially!**

$$\begin{aligned}&\nu_1 < \nu_2, \nu_2 < \nu_1\\&\nu_1 = \nu_2 \qquad (d=2)\end{aligned}$$

$$\begin{aligned}&\nu_1 < \nu_2 < \nu_3, \nu_1 < \nu_3 < \nu_2, \nu_2 < \nu_1 < \nu_3, \nu_2 < \nu_3 < \nu_1, \nu_3 < \nu_1 < \nu_2, \nu_3 < \nu_2 < \nu_1\\&\nu_1 = \nu_2 < \nu_3, \nu_1 = \nu_3 < \nu_2, \nu_2 = \nu_3 < \nu_1\\&\nu_1 < \nu_2 = \nu_3, \nu_2 < \nu_1 = \nu_3, \nu_3 < \nu_1 = \nu_2\\&\nu_1 = \nu_2 = \nu_3 \qquad (d=3)\\&\vdots\end{aligned}$$

- The ordered Bell numbers (or Fubini numbers) count this

Dimension	2	3	4	5	6	7	8	9	10
# of weak orderings	3	13	75	541	4,683	47,293	545,835	7,087,261	102,247,563

Weak Orderings in d Dimensions

- The set of all weak ordering of the threshold crossings partition Ω

$$\mathcal{W} = \left\{ \{ \nu_{p(1)} \preceq_1 \nu_{p(2)} \preceq_2 \dots \preceq_{d-1} \nu_{p(d)} \} : p \text{ is a permutation} \right\}$$

where each $\preceq_j$ is fixed as $<$ or $=$ for each weak ordering

- The approach is to find $\mathbb{P}(W)$ for an arbitrary $W \in \mathcal{W}$ where

$$\begin{aligned} \nu_1 &= \cdots = \nu_{s_1} \\ \nu_{s_1+1} &= \cdots = \nu_{s_2} \\ &\vdots \qquad \vdots \\ \nu_{s_{n-1}+1} &= \cdots = \nu_{s_n}, \end{aligned}$$

and $\nu_1 < \nu_{s_1+1} < \cdots < \nu_{s_{n-1}+1}$

Deriving $\mathbb{P}(W)$ in d Dimensions

- W can be partitioned into disjoint sub-events with $j_1 < j_2 < \cdots < j_n$

$$\begin{aligned} W(q,j) = \{&\nu_1(q_1) = \ldots = \nu_{s_1}(q_{s_1}) = j_1, \\ &\nu_{s_1+1}(q_{s_1+1}) = \cdots = \nu_{s_2}(q_{s_2}) = j_2, \\ &\ldots, \\ &\nu_{s_{n-1}+1}(q_{s_{n-1}+1}) = \ldots = \nu_d(q_d) = j_n\}, \end{aligned}$$

- We then sum over these probabilities

$$\mathbb{P}(W) = \sum_{j_1=1}^{\infty} \sum_{j_2=j_1+1}^{\infty} \cdots \sum_{j_n=j_{n-1}+1}^{\infty} \mathbb{E}\left[\mathbb{1}_{W(q,j)}\right].$$

Operators in d Dimensions

- Let $q = (q_1, q_2, ..., q_d) \in \mathbb{R}^d_{\geq 0}$ and consider the operator

$$\mathcal{H}_q(\cdot)(x) = \mathcal{H}_{q_1} \circ \mathcal{H}_{q_2} \circ \cdots \circ \mathcal{H}_{q_d}(\cdot)(x_1, x_2, ..., x_d)$$

- **Operational calculus** strategy to derive $\mathbb{P}(W)$ for $W \in \mathcal{W}$

$$\mathbb{P}(W) \xrightarrow{\mathcal{H}} \mathbb{Q}(W) \xrightarrow{\text{Assumptions}} \mathbb{Q}(W) \text{ (convenient form) } \xrightarrow{\mathcal{H}^{-1}} \mathbb{P}(W) \text{ (tractable)}$$

Deriving $\mathbb{P}(W)$ in d Dimensions

- The operator applies only to the innermost term by Fubini's Theorem

$$\mathcal{H}_q\left[\mathbb{P}(W)\right](x) = \sum_{j_1=1}^{\infty} \sum_{j_2=j_1+1}^{\infty} \cdots \sum_{j_n=j_{n-1}+1}^{\infty} \mathbb{E}\left[\mathcal{H}_q\left[\mathbb{1}_{W(q,j)}\right](x)\right]$$

Lemma (White [2021a])

For an event $W \in \mathcal{W}$, suppose each $\nu_i = j_k$ for each $i = s_{k-1}+1, s_{k-1}+2, ..., s_k$, then

$$\mathcal{H}_q\left(\mathbb{1}_{W(q,j)}\right)(x) = \prod_{k=1}^{n} \prod_{m=1}^{r_k} \left(z_{\ell_{km}}^{A_{\ell_{km}}^{[j_k-1]}} - z_{\ell_{km}}^{A_{\ell_{km}}^{[j_k]}} \right),$$

where $\ell_{km} = s_{k-1} + m$ and

$$z_k = \begin{cases} e^{-x_k}, & \textit{if } A_k^{[n]} \textit{ is continuous} \\ x_k, & \textit{if } A_k^{[n]} \textit{ is discrete} \end{cases}$$

Use the **independence** the jumps $X^{[k]}$ to split expectation at steps $j_1 - 1, j_1, j_2 - 1, j_2, ..., j_n - 1, j_n$

$$
\begin{aligned}
&\mathcal{H}_q(\mathbb{E}[\mathbb{1}_{W(q,j)}])(x) \\
&= \sum_{j_1=1}^{\infty} \mathbb{E}\left[e^{-x \cdot A^{[j_1-1]}}\right] \mathbb{E}\left[e^{-x^{T_2} \cdot X^{[j_1-1]}} \prod_{i=1}^{r_1} \left(1 - e^{-x_i X_i^{[j_1]}}\right)\right] \\
&\quad \sum_{j_2=j_1+1}^{\infty} \mathbb{E}\left[e^{-x^{T_2} \cdot \left(X^{[j_1+1]} + \dots + X^{[j_2-1]}\right)}\right] \mathbb{E}\left[e^{-x^{T_3} \cdot X^{[j_2]}} \prod_{i=1}^{r_2} \left(1 - e^{-x_{\ell_{2i}} X_{\ell_{2i}}^{[j_2]}}\right)\right] \\
&\quad \times \cdots \\
&\quad \times \sum_{j_n=j_{n-1}+1}^{\infty} \mathbb{E}\left[e^{-x^{T_n} \cdot \left(X^{[j_{n-1}+1]} + \dots + X^{[j_n-1]}\right)}\right] \mathbb{E}\left[\prod_{i=1}^{r_n} \left(1 - e^{-x_{\ell_{ni}} X_{\ell_{ni}}^{[j_n]}}\right)\right]
\end{aligned}
$$

where $S_j = \{k \in \mathbb{N} : s_{j-1} < k \leq s_j\}$ for $j = 1, 2, ..., n$ with $S_{n+1} = \emptyset$ and $T_j = \bigcup_{m=j}^{n} S_m$ and, for $B \subseteq \mathbb{N}_{\leq d}$ and each $i \leq d$, define $x^B = (x_1^B, ..., x_d^B)$ where $x_i^B = \mathbb{1}_B(i) x_i$.

Deriving $\mathbb{P}(W)$ in d Dimensions - Independence - A Lemma

Lemma (White [2021a])

For a vector $y = (y_1, ..., y_d) \in \mathbb{C}^d$,

$$\prod_{k=1}^{d}(1 - y_k) = \sum_{k=0}^{d}(-1)^k f_y(d, k)$$

where

$$f_y(d, k) = \sum_{\substack{N \subseteq \{1,...,d\} \\ |N|=k}} \prod_{j=1}^{k} y_{n_j},$$

where the sum runs over all k*-subsets of* $\{1, ..., d\}$*, each denoted* $N = \{n_1, ..., n_k\}$*, and we define* $f_y(d, 0) = 1$.

Deriving $\mathbb{P}(W)$ in d Dimensions

- Exploit the **identical distribution** to get γ terms
  - Some lead to geometric series
  - Some lead to terms like the lemma
- Sum them and apply the inverse operator!

Theorem (White [2021a])

If each vector x^{T_j} contains at least one component with a positive real part, then for each $W \in \mathcal{W}$,

$$\mathbb{P}(W) = \mathbb{E}\left[\mathbb{1}_W\right] = \mathcal{H}_x^{-1}\left(\prod_{m=1}^{n} \frac{1}{1-\gamma_{T_m}} \sum_{k=0}^{r_m} (-1)^k \sum_{\substack{M \subseteq S_m \\ |M|=k}} \gamma_{T_{m+1} \cup M}\right)(\mathbf{M})$$

where $\gamma_B = \gamma\left(x^B\right)$ *for* $B \in \{1, 2, ..., d\}$

An Application to Stochastic Networks

- A network is under attack
- *i.i.d.* batches of nodes of random size $X_1^{[j]}$ are disabled
- Nodes have *i.i.d.* random weights Y_k with sum

$$X_2^{[j]} = \sum_{k=1}^{X_1^{[j]}} Y_k,$$

- Can we predict the biggest weakness in the network?

Corollary

If node batch sizes are geometrically distributed with parameter p and node weights are exponentially distributed with parameter μ, then

$$\mathbb{P}(\nu_1 < \nu_2) = P(M_1 - 1, \mu M_2) - \frac{e^{-p\mu M_2}}{(1-p)^{M_1-1}} P(M_1 - 1, (1-p)\mu M_2),$$

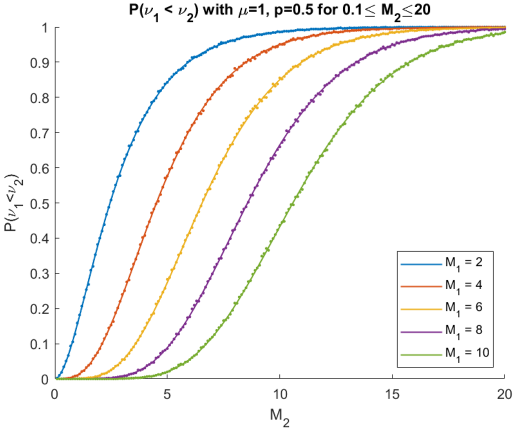
where $P(a,x) = \frac{1}{\Gamma(a)} \int_0^x t^{a-1} e^{-t}\, dt$ is the lower regularized gamma function.

$$\mathbb{P}(\nu_1 > \nu_2) = Q(M_1 - 1, \mu M_2) - (1-p)^{M_1-1} e^{\frac{p\mu M_2}{1-p}} Q\left(M_1 - 1, \frac{\mu M_2}{1-p}\right),$$

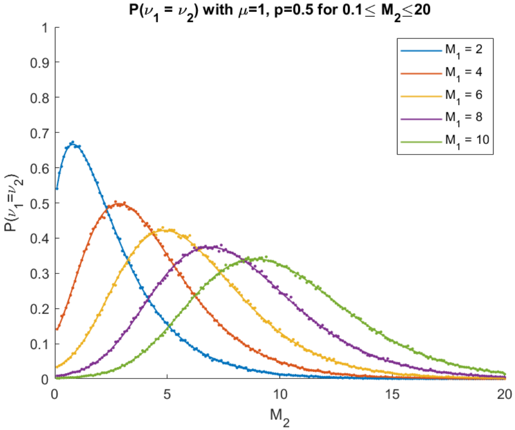
where $Q(a,x) = \frac{1}{\Gamma(a)} \int_x^\infty t^{a-1} e^{-t}\, dt = 1 - P(a,x)$ is the upper regularized gamma function.

$$\mathbb{P}(\nu_1 = \nu_2) = 1 - \mathbb{P}(\nu_1 < \nu_2) - \mathbb{P}(\nu_1 > \nu_2)$$

Predicted Results vs. Monte Carlo Simulations



(a) $\mathbb{P}(\nu_1 < \nu_2)$ with $\mu = 1$, $p = 0.5$



(b) $\mathbb{P}(\nu_1 > \nu_2)$ with $\mu = 1$, $p = 0.5$

Figure: Predicted results and empirical probabilities from 10,000 simulations for stochastic network model

Results for a 2D Exponential Model

Corollary

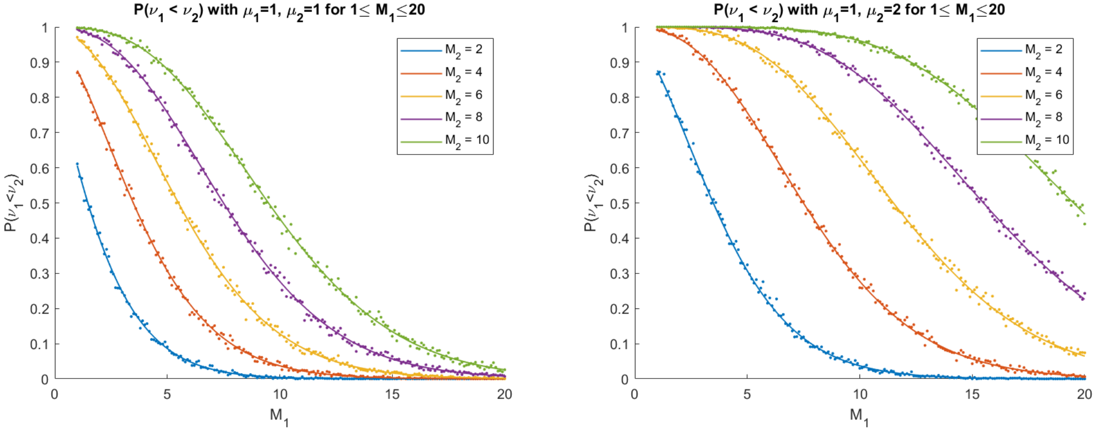
If $X^{[n]} = \left(X_1^{[n]}, X_2^{[n]}\right)$ *is a vector of independent exponential random variables with parameters* μ_1 *and* μ_2 *for each* n*,*

$$\mathbb{P}(\nu_1 < \nu_2) = 1 - e^{-\mu_2 M_2}\left(1 + \sqrt{\mu_1\mu_2 M_2}\int_0^{M_1} \frac{e^{-\mu_1\tau}}{\sqrt{\tau}} I_1\left(2\sqrt{\mu_1\mu_2 M_2\tau}\right)\, d\tau\right),$$

$$\mathbb{P}(\nu_1 > \nu_2) = 1 - e^{-\mu_1 M_1}\left(1 + \sqrt{\mu_1\mu_2 M_1}\int_0^{M_2} \frac{e^{-\mu_2\tau}}{\sqrt{\tau}} I_1\left(2\sqrt{\mu_1\mu_2 M_1\tau}\right)\, d\tau\right),$$

and $\mathbb{P}(\nu_1 = \nu_2) = 1 - \mathbb{P}(\nu_1 < \nu_2) - \mathbb{P}(\nu_1 > \nu_2)$*, where* $I_1(x)$ *is the modified Bessel function of the first kind.*

Predicted Results vs. Monte Carlo Simulations



(a) $\mathbb{P}(\nu_1 < \nu_2)$ with $\mu_1 = \mu_2 = 1$

(b) $\mathbb{P}(\nu_1 < \nu_2)$ with $\mu_1 = 1, \mu_2 = 2$

Figure: Predicted results and empirical probabilities from 10,000 simulations for 2D independent exponential model

Corollary

If $X^{[n]} = \left(X_1^{[n]}, X_2^{[n]}, X_3^{[n]}\right)$ *is a vector of independent exponential random variables with parameters* μ_1, μ_2, *and* μ_3 *for each* n,

$$\begin{aligned}\mathbb{P}(\nu_1 < \nu_2 < \nu_3) =& 1 - e^{-\mu_2 M_2}\left(1 + \sqrt{\mu_1\mu_2 M_2}\int_0^{M_1} \frac{e^{-\mu_1\tau}}{\sqrt{\tau}} I_1\left(2\sqrt{\mu_1\mu_2 M_2\tau}\right)\,d\tau\right) \\ & - e^{-\mu_3 M_3}\left(1 + \sqrt{\mu_2\mu_3 M_3}\int_0^{M_2} \frac{e^{-\mu_2\tau}}{\sqrt{\tau}} I_1\left(2\sqrt{\mu_2\mu_3 M_3\tau}\right)\,d\tau\right) \\ & + e^{-\mu_2 M_2 - \mu_3 M_3} \\ & \times \mathcal{L}_{x_1}^{-1}\left(\frac{e^{\frac{\mu_1\mu_2 M_2}{\mu_1 + x_1}}}{x_1}\int_0^{M_2} \frac{e^{-\frac{\mu_1\mu_2\tau}{\mu_1 + x_1}}}{\sqrt{\tau}} I_1\left(2\sqrt{\frac{\mu_1\mu_2\mu_3 M_3\tau}{\mu_1 + x_1}}\right)\,d\tau\right)(M_1).\end{aligned}$$

Results for a 3D Exponential Model

Corollary

If $X^{[n]} = \left(X_1^{[n]}, X_2^{[n]}, X_3^{[n]}\right)$ *is a vector of independent exponential random variables with parameters* μ_1*,* μ_2*, and* μ_3 *for each* n*,*

$$\begin{aligned}\mathbb{P}(\nu_1 = \nu_2 < \nu_3) = &\left(1 - e^{-\mu_3 M_3}\right) e^{-\mu_1 M_1 - \mu_2 M_2} I_0\left(2\sqrt{\mu_1\mu_2 M_1 M_3}\right) \\ &- e^{-\mu_1 M_1 - \mu_2 M_2 - \mu_3 M_3}\sqrt{\mu_1\mu_2\mu_3 M_3} \\ &\times \mathcal{L}_{x_1}^{-1}\left(\frac{e^{\frac{\mu_1\mu_2 M_2}{x_1}}}{x_1\sqrt{x_1}}\int_0^{M_2}\frac{e^{-\frac{\mu_1\mu_2\tau}{x_1}}}{\sqrt{\tau}} I_1\left(2\sqrt{\frac{\mu_1\mu_2\mu_3 M_3\tau}{x_1}}\right)d\tau\right)(\tau).\end{aligned}$$

Corollary

If $X^{[n]} = \left(X_1^{[n]}, X_2^{[n]}, X_3^{[n]}\right)$ *is a vector of independent exponential random variables with parameters* μ_1, μ_2, *and* μ_3 *for each* n,

$$\begin{aligned}\mathbb{P}(\nu_1 < \nu_2 = \nu_3) =& e^{-\mu_2 M_2 - \mu_3 M_3} I_0\left(2\sqrt{\mu_2\mu_3 M_2 M_3}\right) \\ & - e^{-\mu_2 M_2 - \mu_3 M_3} \mathcal{L}_{x_1}^{-1}\left(\frac{1}{x_1} I_0\left(2\sqrt{\frac{\mu_1\mu_2\mu_3 M_2 M_3}{\mu_1 + x_1}}\right)\right)(M_1).\end{aligned}$$

$$\mathbb{P}(\nu_1 = \nu_2 = \nu_3) = e^{-\mu_1 M_1 - \mu_2 M_2 - \mu_3 M_3} \mathcal{L}_{x_1}^{-1}\left(\frac{1}{x_1} I_0\left(2\sqrt{\frac{\mu_1\mu_2\mu_3 M_2 M_3}{x_1}}\right)\right)(M_1).$$

- The inverse Laplace transforms above that could not be calculated explicitly were computed with the fixed Talbot algorithm developed by Talbot [1979].
- The approach uses some ideas from the framework of Abate and Whitt [2006] and the code is optimised for MATLAB by McClure [2013].
- A similar approach was used to find probabilities in the context of oscillating random walks by Dshalalow and Liew [2006a], but it was minimal
- For each set of parameters, the probabilities are listed in table below ordered as:

$$\nu_1 < \nu_2 < \nu_3,\ \nu_1 < \nu_3 < \nu_2,\ \nu_2 < \nu_1 < \nu_3,\ \nu_2 < \nu_3 < \nu_1,\ \nu_3 < \nu_1 < \nu_2,\ \nu_3 < \nu_2 < \nu_1$$
$$\nu_1 = \nu_2 < \nu_3,\ \nu_1 = \nu_3 < \nu_2,\ \nu_2 = \nu_3 < \nu_1$$
$$\nu_1 < \nu_2 = \nu_3,\ \nu_2 < \nu_1 = \nu_3,\ \nu_3 < \nu_1 = \nu_2$$
$$\nu_1 = \nu_2 = \nu_3$$

Results for a 3D Exponential Model

Parameters		Predicted Probabilities	Errors
(1)	<<	0.125 0.125 0.125 0.125 0.125 0.125	Sum $= 0.003$
$(\mu_1, \mu_2, \mu_3) = (1, 1, 1)$	=<	0.042 0.042 0.042	Max $< 10^{-3}$
$M_1 = M_2 = M_3 = 10$	<=	0.039 0.039 0.039	
	==	0.009	
(2)	<<	0.137 0.137 0.137 0.137 0.137 0.137	Sum $= 0.002$
$(\mu_1, \mu_2, \mu_3) = (1, 1, 1)$	=<	0.030 0.030 0.030	Max $< 10^{-3}$
$M_1 = M_2 = M_3 = 20$	<=	0.029 0.029 0.029	
	==	0.005	
(3)	<<	0.147 0.147 0.147 0.147 0.147 0.147	Sum $= 0.003$
$(\mu_1, \mu_2, \mu_3) = (1, 1, 1)$	=<	0.019 0.019 0.019	Max $< 10^{-3}$
$M_1 = M_2 = M_3 = 50$	<=	0.019 0.019 0.019	
	==	0.002	
(4)	<<	0.153 0.153 0.153 0.153 0.153 0.153	Sum $= 0.002$
$(\mu_1, \mu_2, \mu_3) = (1, 1, 1)$	=<	0.014 0.014 0.014	Max $< 10^{-3}$
$M_1 = M_2 = M_3 = 100$	<=	0.014 0.014 0.014	
	==	0.001	

Results for a 3D Exponential Model

Parameters		Predicted Probabilities	Errors
(1) $(\mu_1, \mu_2, \mu_3) = (1, 1, 1)$ $M_1 = M_2 = M_3 = 10$	$<<$ $=<$ $<=$ $==$	0.125 0.125 0.125 0.125 0.125 0.125 0.042 0.042 0.042 0.039 0.039 0.039 0.009	Sum $= 0.003$ Max $< 10^{-3}$
(5) $(\mu_1, \mu_2, \mu_3) = (0.5, 1, 1)$ $M_1 = M_2 = M_3 = 10$	$<<$ $=<$ $<=$ $==$	0.358 0.358 0.048 0.008 0.048 0.008 0.035 0.035 0.005 0.082 0.007 0.007 0.004	Sum $= 0.002$ Max $< 10^{-3}$
(6) $(\mu_1, \mu_2, \mu_3) = (2, 1, 1)$ $M_1 = M_2 = M_3 = 10$	$<<$ $=<$ $<=$ $==$	0.002 0.002 0.018 0.422 0.018 0.422 0.002 0.002 0.088 0.001 0.011 0.011 0.001	Sum $= 0.002$ Max $< 10^{-3}$
(7) $(\mu_1, \mu_2, \mu_3) = (5, 1, 1)$ $M_1 = M_2 = M_3 = 10$	$<<$ $=<$ $<=$ $==$	0.000 0.000 0.000 0.455 0.000 0.455 0.000 0.000 0.090 0.000 0.000 0.000 0.000	Sum $= 0.002$ Max $< 10^{-3}$

Results for a 3D Exponential Model

Parameters		Predicted Probabilities	Errors
(1) $(\mu_1, \mu_2, \mu_3) = (1, 1, 1)$ $M_1 = M_2 = M_3 = 10$	$<<$ $=<$ $<=$ $==$	0.125 0.125 0.125 0.125 0.125 0.125 0.042 0.042 0.042 0.039 0.039 0.039 0.009	Sum $= 0.003$ Max $< 10^{-3}$
(8) $(\mu_1, \mu_2, \mu_3) = (1, 1, 1)$ $M_1 = 20, M_2 = M_3 = 10$	$<<$ $=<$ $<=$ $==$	0.002 0.002 0.018 0.422 0.018 0.422 0.002 0.002 0.088 0.001 0.011 0.011 0.001	Sum $= 0.001$ Max $< 10^{-3}$
(9) $(\mu_1, \mu_2, \mu_3) = (1, 1, 1)$ $M_1 = 50, M_2 = M_3 = 10$	$<<$ $=<$ $<=$ $==$	0.000 0.000 0.000 0.455 0.000 0.455 0.000 0.000 0.000 0.090 0.000 0.000 0.000 0.000	Sum $= 0.002$ Max $< 10^{-3}$
(10) $(\mu_1, \mu_2, \mu_3) = (1, 1, 1)$ $M_1 = 100, M_2 = M_3 = 10$	$<<$ $=<$ $<=$ $==$	0.000 0.000 0.000 0.455 0.000 0.455 0.000 0.000 0.090 0.000 0.000 0.000 0.000	Sum $< 10^{-4}$ Max $< 10^{-4}$

Insights from 3D Exponential Model

- In parameter sets (1)-(4), we have $\mu_1 = \mu_2 = \mu_3$ and $M_1 = M_2 = M_3$, so in each line of the results for each set, the probabilities are the same since the three dimensions are indistinguishable.
- In parameter sets (1)-(4), M_j's are equal but increasing. Since the process must travel further to cross thresholds while the distribution of the jumps is fixed, simultaneously crossing multiple thresholds becomes less probable.
- Comparing parameter sets (5)-(7) to (1) reveal that increasing a single μ_j decreases the mean jump length in coordinate j so that M_j is likely to be crossed later than others.
- Comparing parameter sets (8)-(10) to (1) reveal that increasing a single M_j has a similar effect as increasing a single μ_j:
  - Parameter set (8) doubles M_1 (doubling the distance to cross M_1) and parameter set (6) halves μ_1 (doubling mean jump length in dimension 1) has the *precisely* same impact on the probabilities.
  - Parameter sets (7) and (9) exhibit an analogous relationship.

Future Work

- Expressions for $\Phi(u) = \mathbb{E}\left[e^{-u \cdot A^{[\nu]}}\right]$ have been derived by brute force for $d \leq 4$ similar setting by Dshalalow and his collaborators
- Work algorithmic solutions in White [2015] showed automating this poses computational problems
- The result for W provides a path to representing an arbitrary Φ_W in a manageable formula.
- Ongoing work White [2021b] seeks to either confirm a conjecture from Dshalalow and Liew [2006c] or correct it by deriving an expression for

$$\Phi(u) = \sum_{W \in \mathcal{W}} \Phi_W(u)$$

through a recursive approach exploiting patterns in the Φ_W expressions.

Future Work

- Prior studies Dshalalow and Liew [2006b], Dshalalow and White [2013, 2016] have considered (continuous-time) marked random walks through similar approaches:

$$\{A(t) : t \geq 0\}$$

with multidimensional jump times forming a renewal process $\{\tau_n\}$ for $d \leq 4$

- This work can be adapted to find a joint LST for the position and time of the exit of such a process,

$$\mathbb{E}\left[e^{-u \cdot A(\tau_\nu) - \theta \tau_\nu}\right]$$

to get means, moments, and distributions of exit time and position.

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