1 Highlights (§14.3-14.5)

- Disclaimer: This is NOT a complete list of what you need to understand. Any material in the sections may appear on tests.
- Partial derivatives with respect to a variable x – simply treat other variables as constants and calculate derivatives as in Calc 1.
- Partial differential equations have solutions that are *functions* satisfying some condition on their partial derivatives.
  - For example if $u_{xx} + u_{yy} = 0$ on the whole domain, function u is a solution to Laplace's equation.
- We discussed tangent *lines* previously for space curves, but 3D surfaces $f(x, y)$ have multiple tangent lines along different directions.
  - The tangent plane is the plane including the tangent lines along the positive x and y directions, with slopes determined by $f_x(x_0, y_0)$ and $f_y(x_0, y_0)$ at point (x_0, y_0). See equation 2 in §14.4.
- Linear approximations allow us to estimate the value of a function near a specified point in a computationally efficient way.
- The chain rule largely works as it does in Calc 1, see examples in §14.5.
- Pay particular attention to the implicit function theorem (6-7) in §14.5 and understand when and how to use it

2 Formulas

- Tangent plane to $z = f(x, y)$ at point (x_0, y_0, z_0) (if f has continuous partial derivatives):

$$z - z_0 = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$$

- Linear approximation to f at (a, b):

$$L(x, y) = f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b)$$

- (Implicit differentiation) If F is defined within a sphere containing (a, b, c), where $F(a, b, c) = 0, F_z(a, b, c) \neq 0$, and F_x, F_y, F_z are continuous inside the sphere, then $F(x, y, z) = 0$ defines z as a function of x, y near (a, b, c) and F is differentiable, then

$$\frac{\partial z}{\partial x} = -\frac{F_x}{F_z}$$
$$\frac{\partial z}{\partial y} = -\frac{F_y}{F_z}$$

3 Problems

Example 1 (§14.3): Determine whether $u = \sin(x - at) + \ln(x + at)$ is a solution to the wave equation $u_{tt} = a^2 u_{xx}$.

To solve this, we simply need to find the required partial derivatives and see if they satisfy the wave equation for all possible inputs (x, t):

$$\begin{aligned}
u_x(x,t) &= \cos(x - at) + \frac{1}{x + at} \\
u_{xx}(x,t) &= -\sin(x - at) - \frac{1}{(x + at)^2} \\
u_t(x,t) &= \cos(x - at)(-a) + \frac{1}{x + at}(a) \\
&= -a\cos(x - at) + \frac{a}{x + at} \\
u_{tt}(x,t) &= -a^2 \sin(x - at) - \frac{a^2}{(x + at)^2}
\end{aligned}$$

Then we have

$$a^2 u_{xx}(x,t) = -a^2 \sin(x - at) - \frac{a^2}{(x + at)^2} = u_{tt}(x,t)$$

Thus, u is a solution to the wave equation.

Example 2 (§14.4): Find the linear approximation of the function $f(x, y) = 1 - xy\cos(\pi y)$ at $(1, 1)$ and use it to approximate $f(1.02, 0.97)$.

The linear approximation of f at (a, b) is $L(x, y) = f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b)$, so we need the partial derivatives:

$$\begin{aligned}
f_x &= -y\cos(\pi y) \\
f_y &= -x\cos(\pi y) + xy\sin(\pi y)\pi
\end{aligned}$$

Then,

$$\begin{aligned}
L(x, y) &= f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b) \\
&= 1 - ab\cos(\pi b) - b\cos(\pi b)(x - a) + (-a\cos(\pi b) + ab\sin(\pi b)\pi)(y - b) \\
&= 1 - \cos(\pi) - \cos(\pi)(x - 1) + (-\cos(\pi) + \pi\sin(\pi))(y - 1) \\
&= 2 + (x - 1) + (1 + 0)(y - 1) \\
&= x + y
\end{aligned}$$

$f(1.02, 0.97) \approx 1.02 + 0.97 = 1.99$

While this example may seem useless since we can easily find $f(1.02, 0.97)$ with a calculator, the point is to understand this concept for functions f that are not so easy to calculate numerically, but whose partial derivatives are easy to find or estimate.

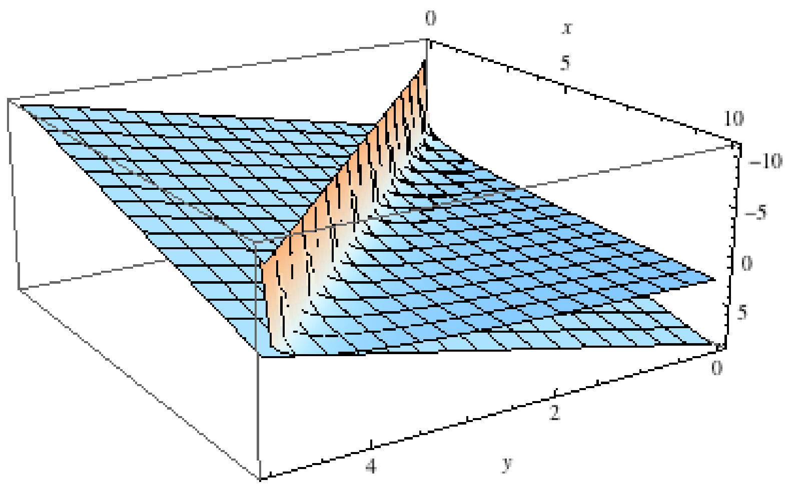
Example 3 (§14.4): Find the equation of the tangent plane to the surface $f(x, y) = \ln(x - 2y)$ at point $(3, 1, 0)$.

First, we need the partial derivatives.

$$f_x = \frac{1}{x - 2y}$$
$$f_y = \frac{1}{x - 2y}(-2) = \frac{-2}{x - 2y}$$

Then the formula of the tangent plane at the given point is

$$\begin{aligned} z &= f_x(3, 1)(x - 3) + f_y(3, 1)(y - 1) \\ &= (x - 3) - 2(y - 1) \end{aligned}$$



Example 4 (§14.5): Find the equation of the tangent plane to the surface $e^z = xyz$ at point $\left(\frac{1}{\ln(2)}, 2, \ln(2)\right)$

Since it is not clear how to write the function in the form $z = f(x, y)$, we will need to use implicit differentiation on the function $F(x, y, z) = xyz - e^z = 0$. We still need f_x and f_y. By the Implicit Function Theorem (7), we find:

$$f_x = \frac{\partial z}{\partial x} = -\frac{F_x}{F_z} = -\frac{yz}{xy - e^z}$$
$$f_y = \frac{\partial z}{\partial y} = -\frac{F_y}{F_z} = -\frac{xz}{xy - e^z}$$

Thus, at point $\left(\frac{1}{\ln(2)}, 2, \ln(2)\right)$, we have

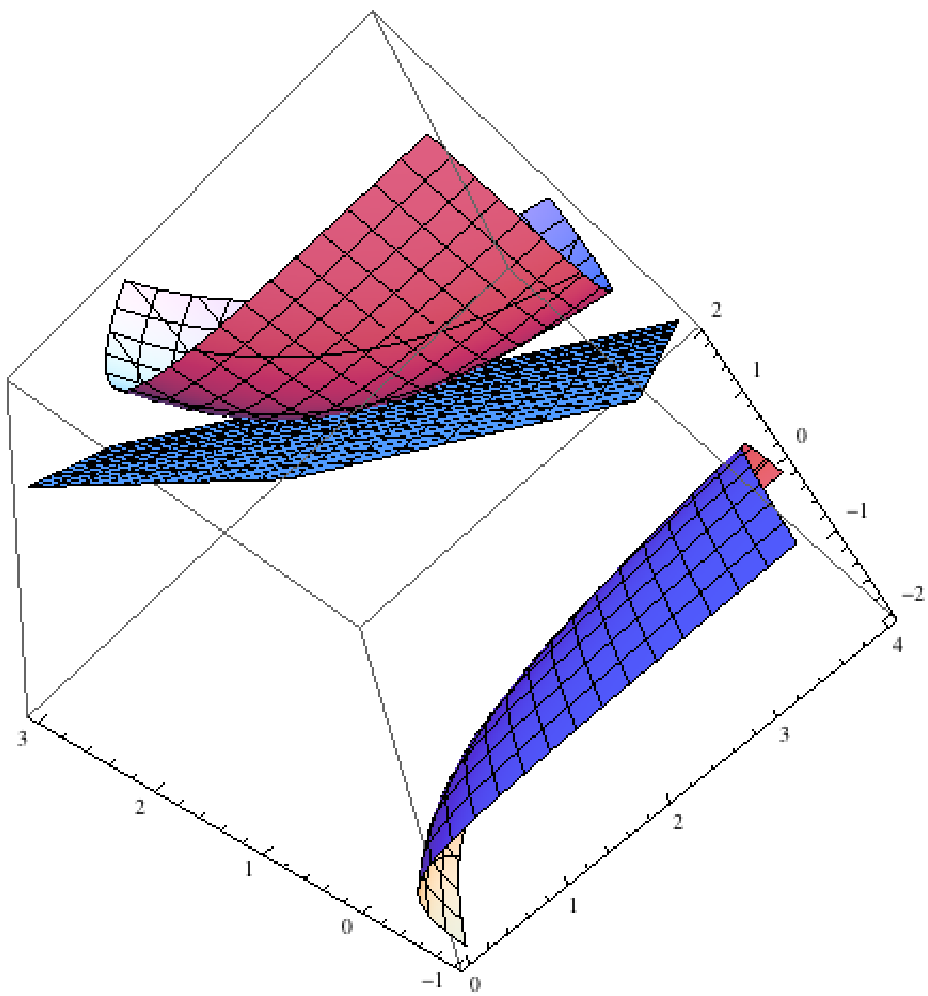
$$f_x\left(\frac{1}{\ln(2)}, 2, \ln(2)\right) = -\frac{2\ln(2)}{\frac{2}{\ln(2)} - 2} = \frac{-2\ln(2)^2}{2 - 2\ln(2)}$$

$$f_y\left(\frac{1}{\ln(2)}, 2, \ln(2)\right) = -\frac{1}{\frac{2}{\ln(2)} - 2} = \frac{-\ln(2)}{2 - 2\ln(2)}$$

Using formula (2) in §14.4, we have the plane

$$z - \ln(2) = \frac{-2\ln(2)^2}{2 - 2\ln(2)}\left(x - \frac{1}{\ln(2)}\right) - \frac{\ln(2)}{2 - 2\ln(2)}(y - 2)$$

Since this function has discontinuities, you will see disconnected portions of the surface in the image along with the tangent plane at the given point.



$z = \sin(x)\cos(y)$, $0 \le x \le 4\pi$, $0 \le y \le 4\pi$. At which points (x, y) are the tangent planes parallel to the xy-plane?

For which points (x, y) are the tangent planes to the sphere of radius 3 centered at $(1, -2, 2)$ perpendicular to the xy-plane?