1 Highlights (§14.6-14.7)

- Disclaimer: This is NOT a complete list of what you need to understand. Any material in the sections may appear on tests.
- The gradient vector is a vector function $\nabla f = \langle f_x, f_y, f_z \rangle$ (it is defined analogously for functions of 2 variables).
- In §14.6, we extend these ideas of partial derivatives to directional derivatives, which represent the rate of change of the function in the direction of a unit vector $\mathbf{u} = \langle a, b, c \rangle$,

$$D_{\mathbf{u}} f(x, y, z) = \mathbf{u} \cdot \nabla f(x, y, z) = \mathbf{u} \cdot \langle f_x(x, y, z), f_y(x, y, z), f_z(x, y, z) \rangle \quad (1)$$

- The maximum directional derivative $D_{\mathbf{u}} f(x, y, z)$ occurs when $\mathbf{u}$ has the same direction as $\nabla f(x, y, z)$ and has maximum value $|\nabla f(x, y, z)|$ (Theorem 15).
- Any point (a, b) with $f_x(a, b) = f_y(a, b) = 0$ is called a critical point.
- The Second Derivatives Test allows us to classify many critical points as local minimums, maximums, or saddle points.
- If f is continuous on a closed, bounded set D in $\mathbb{R}^2$, then f attains a minimum and maximum value on D.
- To find the absolute minimum or maximum of f on D,
  1. Find the values of f at all critical points of f in D.
  2. Find the maximum and minimum values of f on the boundary of D.
  3. The absolute minimum (maximum) occurs where f is smallest (largest) at the critical points or along the boundary.

2 Problems

Example 1 (§14.2, #18): Evaluate $\lim\limits_{(x,y)\to(0,0)} \frac{xy^4}{x^2+y^8}$ or show it does not exist.

First, we calculate limits along curves and see if we can show the limit does not exist. If we cannot do this, we should pursue the squeeze theorem to prove the limit exists.

A good curve will often cancel the variables in the numerator and leave a nonzero number in the denominator after the limit. Next, we should notice that taking

x to be some multiple of y^4 will allow us to cancel the y's, so:

$$\begin{aligned}\lim_{(my^4,y)\to(0,0)} \frac{my^4y^4}{m^2y^8+y^8} &= \lim_{(my^4,y)\to(0,0)} \frac{my^8}{(m^2+1)y^8} \\ &= \lim_{(my^4,y)\to(0,0)} \frac{m}{m^2+1} = \frac{m}{m^2+1}\end{aligned}$$

Since we could take any multiple m, we see that the limit along $x = y^4$ is $\frac{1}{2}$, while the limit along $x = 2y^4$ is $\frac{2}{5}$, so the limits along these two curves are different, so the limit $\lim_{(x,y)\to(0,0)} \frac{xy^4}{x^2+y^8}$ does not exist.

Example 2 (§14.6, #25): Find the maximum rate of change of $f(x, y, z) = \sqrt{x^2 + y^2 + z^2}$ at the point $(3, 6, -2)$ and the direction in which it occurs.

Theorem 15 tells us that the maximum rate of change is $|\nabla f(3, 6, -2)|$ and in the direction of $\nabla f(3, 6, -2)$, so first, we calculate the partial derivatives at $(3, 6, -2)$,

$$\begin{aligned} f_x(3,6,-2) &= \left.\frac{2x}{2\sqrt{x^2+y^2+z^2}}\right|_{(3,6,-2)} = \left.\frac{x}{\sqrt{x^2+y^2+z^2}}\right|_{(3,6,-2)} = \frac{3}{7} \\ f_y(3,6,-2) &= \left.\frac{y}{\sqrt{x^2+y^2+z^2}}\right|_{(3,6,-2)} = \frac{6}{7} \\ f_z(3,6,-2) &= \left.\frac{z}{\sqrt{x^2+y^2+z^2}}\right|_{(3,6,-2)} = -\frac{2}{7} \end{aligned}$$

Then the maximum rate of change occurs in the direction $\nabla f(3, 6, -2) = \left\langle \frac{3}{7}, \frac{6}{7}, -\frac{2}{7} \right\rangle$, and its value is $|\nabla f(3, 6, -2)| = \sqrt{\frac{9}{49} + \frac{36}{49} + \frac{4}{49}} = 1$. (Note that any multiple of $\nabla f(3, 6, -2)$ would also give a valid direction.)

Example 3 (§14.6, #45): Find the equations of the tangent plane and normal line to the surface $x + y + z = e^{xyz}$ at $(0, 0, 1)$.

§14.6 provides a strategy aside from implicit differentiation to find tangent planes when we cannot easily write the equation in the form $z = f(x, y)$. For $F(x, y, z) = k$ for some constant k, the tangent plane can be written

$$F_x(x_0, y_0, z_0)(x - x_0) + F_y(x_0, y_0, z_0)(y - y_0) + F_z(x_0, y_0, z_0)(z - z_0) = 0 \quad (2)$$

First we set up $F(x, y, z) = e^{xyz} - x - y - z = 0$ and find the partial derivatives.

$$\begin{aligned} F_x(x,y,z) &= yze^{xyz} - 1 \\ F_y(x,y,z) &= xze^{xyz} - 1 \\ F_z(x,y,z) &= xye^{xyz} - 1 \end{aligned}$$

Plugging in $(x_0, y_0, z_0) = (0, 0, 1)$ and using formula (2),

$$\begin{aligned} F_x(0,0,1)x + F_y(0,0,1)y + F_z(0,0,1)(z-1) &= 0 \\ -x - y - (z-1) &= 0 \end{aligned}$$

The normal line is orthogonal to the tangent plane and passes through $(0, 0, 1)$, so the symmetric equations of the normal line are

$$\frac{x}{F_x(0,0,1)} = \frac{y}{F_y(0,0,1)} = \frac{z-1}{F_z(0,0,1)}$$
$$-x = -y = -z + 1$$

Example 4 (§14.6, #46): Find the equations of the tangent plane and normal line to the surface $x^4 + y^4 + z^4 = 3x^2y^2z^2$ at $(1, 1, 1)$.

First, we set up $F(x, y, z) = x^4 + y^4 + z^4 - 3x^2y^2z^2 = 0$, and find the derivatives:

$$\begin{aligned} F_x(x,y,z) &= 4x^3 - 6xy^2z^2 \\ F_y(x,y,z) &= 4y^3 - 6x^2yz^2 \\ F_z(x,y,z) &= 4z^3 - 6x^2y^2z \end{aligned}$$

Plugging in (x_0, y_0, z_0), and using formula (2), we find the tangent plane:

$$\begin{aligned} F_x(1,1,1)(x-1) + F_y(1,1,1)(y-1) + F_z(1,1,1)(z-1) &= 0 \\ -2(x-1) - 2(y-1) - 2(z-1) &= 0 \end{aligned}$$

The normal line is orthogonal to the tangent plane and passes through $(1, 1, 1)$, so the symmetric equations of the normal line are

$$\frac{x - x_0}{F_x(x_0,y_0,z_0)} = \frac{y - y_0}{F_y(x_0,y_0,z_0)} = \frac{z - z_0}{F_z(x_0,y_0,z_0)}$$
$$-\frac{x-1}{2} = -\frac{y-1}{2} = -\frac{z-1}{2}$$

Example 5 (§14,7, #11): Find the local maximum and minimum values and saddle point(s) of the function $f(x, y) = x^3 - 12xy + 8y^3$.

The critical points are points (a, b) such that $f_x(a, b) = f_y(a, b) = 0$, which we need to find, so we will calculate the partial derivatives and set them equal to 0, and solve.

$$\begin{aligned} f_x(x,y) &= 3x^2 - 12y = 0 \\ f_y(x,y) &= -12x + 24y^2 = 0 \end{aligned}$$

Solving the first equation gives $y = \frac{x^2}{4}$, plugging this into the second equation, we find

$$\begin{aligned} f_y\left(x, \frac{x^2}{4}\right) = -12x + 24\frac{x^4}{16} &= 0 \\ x - 2\frac{x^4}{16} &= 0 \\ x\left(1 - \frac{x^3}{8}\right) &= 0 \end{aligned}$$

Solving this yields $x = 0$ and $x = 2$, then we can find the y values for each since $y = \frac{x^2}{4}$, so the critical points are $(0, 0)$ and $(2, 1)$. Next, we use the Second Derivative Test to classify these critical points.

$$\begin{aligned} f_{xx}(x, y) &= 6x \\ f_{yy}(x, y) &= 48y \\ f_{xy}(x, y) &= -12 \end{aligned}$$

Then we have

$$\begin{aligned} D(0, 0) &= (0)(0) - 144 < 0 \\ D(2, 1) &= (12)(48) - 144 = 432 > 0 \\ f_{xx}(2, 1) &= 12 > 0 \end{aligned}$$

Thus, $(0, 0)$ is a saddle point and $(2, 1)$ is a local minimum with value $f(2, 1) = -8$.

Example 6 (§14.7, #30): Find the absolute maximum and minimum of $f(x, y) = x + y - xy$ on the closed triangular region enclosed by $(0, 0)$, $(0, 2)$, and $(4, 0)$.

First, we will seek the local minimum and maximum points by solving for x and y such that $f_x(x, y) = f_y(x, y) = 0$. Clearly, if $f_x(x, y) = 1 - y$ and $f_y(x, y) = 1 - x$, then both are 0 only at $x = y = 1$, with $f(1, 1) = 1$.

Next, we will seek the maximum and minimum of the function along the boundary of the triangle. First, we have the line segment $(0, y)$ for $0 \leq y \leq 2$, where we have $f(0, y) = y$, which is maximized at 2 for $y = 2$ and minimized at 0 for $y = 0$.

Second, we have the line segment $(x, 0)$ for $0 \leq x \leq 4$, along which $f(x, 0) = x$, which is maximized at 4 for $x = 4$ (the highest so far) and minimized at 0 for $x = 0$.

Lastly, we have the third side of the triangle, $y = \frac{-x}{2} + 2$, where we have

$$g(x) = f\left(x, \frac{-x}{2} + 2\right) = x - \frac{x}{2} + 2 + \frac{x^2}{2} - 2x = \frac{x^2}{2} - \frac{3x}{2} + 2$$

We maximize this function of 1 variable where $0 \leq x \leq 2$ by finding where the derivative is zero and compare to the endpoints:

$$\begin{aligned} g'(x) = x - \frac{3}{2} &= 0 \\ x &= \frac{3}{2} \end{aligned}$$

Then, at this point we have $f\left(\frac{3}{2}, \frac{5}{4}\right) = \frac{3}{2} + \frac{5}{4} - \frac{15}{8} = \frac{7}{8}$, so the maximum on the third side of the triangle is $f(4, 0) = 4$. Thus, the absolute maximum occurs at $(4, 0)$ and the absolute minimum occurs at $(0, 0)$.

Example 7 (§14.7, #35): Find the absolute minimum and absolute maximum of $f(x, y) = 2x^3 + y^4$ on the domain $D = \{(x, y) | x^2 + y^2 \leq 1\}$

When we're given a domain, we know the absolute minimum and maximum will occur either at critical points or along the boundary of the domain. First, we should find the critical point(s):

$$\begin{aligned} f_x(x, y) &= 6x^2 \\ f_y(x, y) &= 4y^3 \end{aligned}$$

These are both zero only at $(0, 0)$, so this is our only critical point, where $f(0, 0) = 0$.

Next, we find the maximum and minimum along the boundary by reducing f to a function of one variable and using Calculus 1 techniques. This domain is the filled-in circle centered at $(0, 0)$ with radius 1, so our boundary is the circle $x^2 + y^2 = 1$, and we can use $y^2 = 1 - x^2$ to define our function conveniently:

$$\begin{aligned} g(x) = f(x, y) = 2x^3 + y^4 = 2x^3 + (1 - x^2)^2 = x^4 + 2x^3 - 2x^2 + 1 \\ g'(x) = 4x^3 + 6x^2 - 4x = 2x(2x^2 + 3x - 2) = 2x(2x - 1)(x + 2) \end{aligned}$$

Setting $g'(x) = 0$ and solving for x, we find the critical points (in the 2D sense) along the boundary to occur at $x = 0$, $x = -2$, and $x = \frac{1}{2}$. We can immediately disregard $x = -2$ because $-1 \leq x \leq 1$ on this circle, then the points remaining are $(0, \pm 1)$ and $\left(\frac{1}{2}, \pm\frac{\sqrt{3}}{2}\right)$. The value of the function at each is $g(0) = f(0, \pm 1) = 1$ and $g\left(\frac{1}{2}\right) = f\left(\frac{1}{2}, \pm\frac{\sqrt{3}}{2}\right) = \frac{13}{16}$

We also need to test the points at the endpoints of the boundary, i.e. where $x = -1$ and $x = 1$. $g(-1) = f(-1, 0) = -2$ and $g(1) = f(1, 0) = 2$.

Thus, on the domain D, the absolute minimum is $f(-1, 0) = -2$ and the abslute maximum is $f(1, 0) = 2$.

Example 8 (§14.7, #43): Find three positive numbers whose sum is 100 and whose product is a maximum.

Since $x + y + z = 100$, and we want to maximize xyz. First, we want to reduce this to a function of 2 variables by noting $z = 100 - x - y$, then we seek to maximize $f(x, y) = xy(100 - x - y)$. Next, find the first partial derivatives:

$$\begin{aligned} f_x(x,y) &= y(100 - x - y) - xy = 100y - y^2 - 2xy \\ f_y(x,y) &= 100x - x^2 - 2xy \end{aligned}$$

Setting $f_x(x, y) = 0$ implies $y(100 - y - 2x) = 0$, so $y = 0$ or $y = 100 - 2x$, but $y = 0$ would not satisfy the condition that y is positive. Then, we plug in $y = 100 - 2x$ into the y derivative and see what values of x will make $f_y(x, 100 - 2x) = 0$. We have

$$\begin{aligned} f_y(x, 100-2x) = 100x - x^2 - 2x(100 - 2x) &= 0 \\ 100x - x^2 - 200x + 4x^2 &= 0 \\ 3x^2 - 100x &= 0 \\ 3x^2 &= 100x \\ 3x &= 100 \\ x &= \frac{100}{3} \end{aligned}$$

Then $y = 100 - 2\left(\frac{100}{3}\right) = \frac{100}{3}$. We will now use the Second Derivative Test to see if this point is a local maximum. $f_{xx}(x, y) = -2y$, $f_{yy}(x, y) = -2x$, and $f_{xy}(x, y) = 100 - 2y - 2x$, then

$$\begin{aligned} D\left(\frac{100}{3}, \frac{100}{3}\right) &= f_{xx}\left(\frac{100}{3}, \frac{100}{3}\right) f_{yy}\left(\frac{100}{3}, \frac{100}{3}\right) - f_{xy}\left(\frac{100}{3}, \frac{100}{3}\right)^2 \\ &= -\frac{200}{3} \cdot \frac{-200}{3} - \left(\frac{100}{3}\right)^2 = \frac{40000}{9} - \frac{10000}{9} \\ &= \frac{10000}{3} > 0 \end{aligned}$$

$f_{xx}\left(\frac{100}{3}, \frac{100}{3}\right) < 0$, then $\left(\frac{100}{3}, \frac{100}{3}\right)$ is a local maximum while the boundary points would have one of the variables equal to zero, which yields a zero product, thus this point must be the maximum: $x = y = z = \frac{100}{3}$.

Example 9 (§14.7, #45) Find the maximum volume of a rectangular box that is inscribed in a sphere of radius r.

For convenience, let's center the sphere at $(0,0,0)$, then we have $x^2+y^2+z^2=r^2$. Clearly, the box should touch the sphere on each corner, or else translation or expansion would make it larger while still inside the sphere, the corners of our box must satisfy this formula.

Centering the box at $(0,0,0)$ and orienting it such that its edges are parallel to the coordinate axes is simplest (since rotating it cannot add volume, we may do this), then the dimensions will be $L=2x$, $W=2y$, $D=2z=2\sqrt{r^2-x^2-y^2}$, then the volume can be written $V(x,y)=8xy\sqrt{r^2-x^2-y^2}$, and this is what we must maximize.

$$\begin{aligned} V_x(x,y) &= 8y\sqrt{r^2-x^2-y^2}+\frac{8xy(-2x)}{2\sqrt{r^2-x^2-y^2}} \\ &= \frac{8y(r^2-x^2-y^2)-8x^2y}{\sqrt{r^2-x^2-y^2}} \end{aligned}$$

Then, setting it equal to 0,

$$\begin{aligned} V_x(x,y) = \frac{8y(r^2-x^2-y^2)-8x^2y}{\sqrt{r^2-x^2-y^2}} &= 0 \\ \frac{8y(r^2-2x^2-y^2)}{\sqrt{r^2-x^2-y^2}} &= 0 \\ r^2-2x^2-y^2 &= 0 \\ r^2 = 2x^2+y^2 &= 0 \end{aligned}$$

Similarly, setting $V_y(x,y)=0$ yields $r^2=x^2+2y^2$, then $2x^2+y^2=x^2+2y^2$, which implies $x=y$, then we find

$$\begin{aligned} r^2 &= x^2+2x^2=3x^2 \\ \frac{r}{\sqrt{3}} &= x=y \end{aligned}$$

This is the only critical point and there must be a maximum given the geometric nature of the problem, and so $V\left(\frac{r}{\sqrt{3}},\frac{r}{\sqrt{3}}\right)=8\frac{r^2}{3}\sqrt{r^2-\frac{r^2}{3}-\frac{r^2}{3}}=\frac{8r^3}{3\sqrt{3}}$