Stochastic Analysis of Strategic Networks

Ryan White

Florida Institute of Technology

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Papers

We follow the ideas of the papers

- J. H. Dshalalow and R. White. On Reliability of Stochastic Networks. *Neural, Parallel, and Scientific Computations*, 21 (2013) 141-160
- J. H. Dshalalow and R. White. On Strategic Defense in Stochastic Networks. *Stochastic Analysis and Applications*, accepted for publication (2014)

An Initial Model

On a probability space $(\Omega, \mathcal{F}(\Omega), P)$,

$\{t_1, t_2, ...\}$ – point process of attack times

n_k – *iid* number of nodes lost at t_k with probability-generating function (PGF) $g(z) = E\left[z^{n_1}\right]$.

w_{jk} – *iid* weight per node with Laplace-Stieltjes transform (LST) $l(u) = E\left[e^{-uw_{11}}\right]$

$w_k = \sum_{j=1}^{n_k} w_{jk}$ – total weight lost at t_k

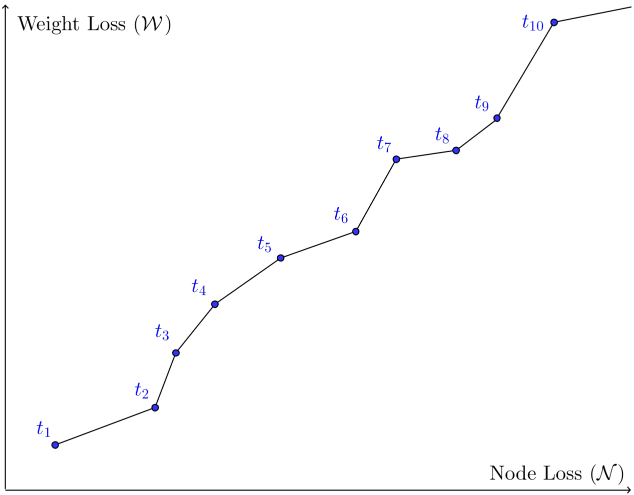
The process is modeled by a marked Poisson random measure of rate λ:

- $\eta(B) = \mathcal{N} \otimes \mathcal{W}(B) = \sum_{k=1}^{\infty} (n_k, w_k)\varepsilon_{t_k}(B)$

for a Borel set $B \in \mathcal{B}(\mathbb{R}_{\geq 0})$.

An Initial Model

We consider the monotone increasing process $\eta([0, t])$ on $\mathbb{Z}_{\geq 0} \times \mathbb{R}_{\geq 0}$



Delayed Observation

While viewing the process in real-time would be helpful, we consider that it is only possible to observe the process upon a third-party renewal process

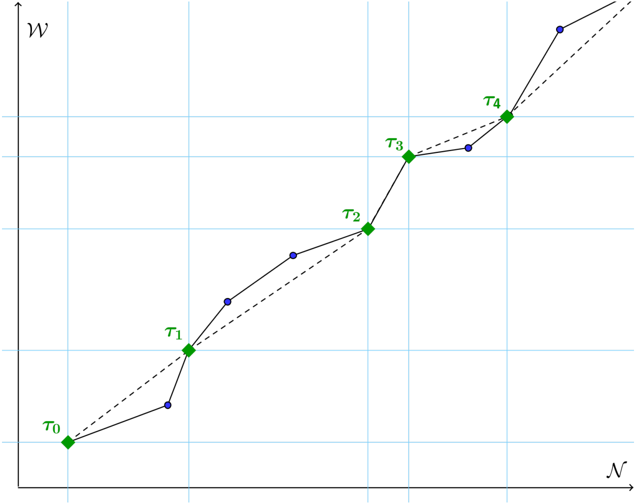
- $\mathcal{T} = \sum_{n=0}^{\infty} \varepsilon_{\tau_n}$
  - $\Delta_k = \tau_k - \tau_{k-1}$ are *iid* with LST $L(\theta)$.

Our process of interest becomes the embedded process

- $Z := \mathcal{X} \otimes \mathcal{Y} \otimes \mathcal{T} = \sum_{n=0}^{\infty} \eta((\tau_{n-1}, \tau_n])\varepsilon_{\tau_n}$
  - $(N_n, W_n) := Z([0, \tau_n])$, the value of the process at τ_n

Delayed Observation

We consider the (green) embedded process



Observed Threshold Crossings

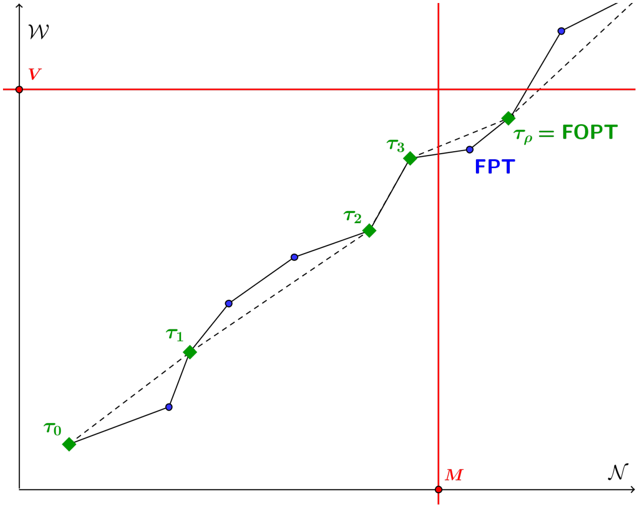
Threshold crossings by the **embedded process** will be of interest.

Each component of the process has a threshold: M for nodes, V for weights, **exit index**

- $\boxed{\rho = \min\{n : N_n > M \text{ or } W_n > V\}}$

The **first observed passage time** (FOPT) is τ_ρ, which may not coincide with the first passage time of the underlying process $\eta([0, t])$

Observed Threshold Crossings



Functional of Interest

We are interested in a joint functional of the values of the components of the process and times upon the pre-FOPT ($\tau_{\rho-1}$) and the FOPT (τ_ρ):

$$\Phi = \Phi(\alpha_0, \alpha, \beta_0, \beta, h_0, h) = E\left[{\alpha_0}^{N_{\rho-1}} \alpha^{N_\rho} e^{-\beta_0 W_{\rho-1} - \beta W_\rho} e^{-h_0 \tau_{\rho-1} - h\tau_\rho}\right]$$

which is helpful because we can find marginal transforms, such as

- $\Phi(1, \alpha, 0, 0, 0, 0) = E\left[\alpha^{N_\rho}\right]$
- $\Phi(1, 1, \beta_0, 0, 0, 0) = E\left[e^{-\beta_0 W_{\rho-1}}\right]$
- $\Phi(1, 1, 0, 0, 0, h) = E\left[e^{-h\tau_\rho}\right]$

which lead to moments and distributions of components of the process upon $\tau_{\rho-1}$ and τ_ρ.

Strategy for finding Φ

We introduce an operator,

$$\mathcal{D}_{pq} = \mathcal{L}\mathcal{C}_q \circ D_p$$

where

$$D_p\{f(p)\}(x) = \sum_{p=0}^{\infty} x^p f(p)(1-x), \|x\| < 1$$

for a sequence $f(p)$, where D has an inverse which can restore f:

$$\mathcal{D}_x^k(\cdot) = \lim_{x \to 0} \frac{1}{k!} \frac{\partial^k}{\partial x^k} \left[\frac{1}{1-x} \cdot \right], \ k \in \mathbb{Z}_{\geq 0}$$
$$\mathcal{D}_x^k(D_p\{f(p)\}(x)) = f(k), k \in \mathbb{Z}_{\geq 0}$$

Strategy for Finding Φ

The inverse to the above operator as applied to some function φ for some thresholds p and q is

$$\mathcal{D}_{xw}^{-1}\left(\varphi(x,w)\right)(p,q) = \mathcal{L}_w^{-1}\left(\frac{1}{w}\mathcal{D}_x^p\left(\varphi(x,w)\right)\right)(q)$$

We apply $\mathcal{D}$ to transform $\Phi \to \Psi$, simplify under some assumptions on the process, and then apply the inverse transform $\mathcal{D}^{-1}$ to return $\Psi \to \Phi$:

$$\Phi \xrightarrow{\mathcal{D}} \Psi \xrightarrow{\text{Assumptions on the process}} \Psi \text{ (explicit)} \xrightarrow{\mathcal{D}^{-1}} \Phi \text{ (tractable)}$$

Results for a Special Case

To demonstrate that tractable results can be derived from this, we will consider a special case where

1. $\Delta_k \in [\text{Exponential}(\mu)] \implies L(\theta) = \frac{\mu}{\mu+\theta}$
2. $n_k \in [\text{Geometric}(a)] \implies g(z) = \frac{az}{1-bz}, (b = 1 - a)$
3. $w_{jk} \in [\text{Exponential}(\xi)] \implies l(u) = \frac{\xi}{\xi+u}$

Theorem 1

$$\Phi = \Phi\left(1, z, 0, v, 0, \theta\right) = E\left[z^{N_\rho} e^{-vW_\rho} e^{-\theta\tau_\rho}\right]$$

$$= 1 - \left(1 - \frac{\mu}{\mu+\theta+\lambda}\frac{v+\xi(1-bz)}{v+\xi(1-c_2 z)}\right)$$

$$\times\left(1 + \frac{b\mu}{\lambda+b\theta} + \frac{a\lambda\mu}{\left(\lambda+b\theta\right)\left(\lambda+\theta\right)}\phi\left(z,v,\theta\right)\right),$$

$$\phi(z,v,\theta) = \frac{v+\xi}{v+\xi(1-c_1 z)} - \frac{c_1 z\xi \boxed{Q(M-1, c_1 z\xi V)} e^{-(v+\xi(1-c_1 z))V}}{v+\xi(1-c_1 z)}$$

$$- \frac{(c_1 z\xi)^M \boxed{P(M-1, (\xi+v)V)}}{(v+\xi(1-c_1 z))(\xi+v)^{M-1}},$$

$$c_1 = \frac{\lambda+b\theta}{\lambda+\theta}, \qquad c_2 = \frac{\lambda+b(\mu+\theta)}{\lambda+\mu+\theta},$$

and $Q(x,y) = \frac{\Gamma(x,y)}{\Gamma(x)}$ is the lower regularized gamma function.

Corollaries: Marginal Transforms

$$\Phi(1, z, 0, 0, 0, 0) = E\left[z^{N_\rho}\right] = \frac{zQ(M-1, z\xi V)e^{-\xi(1-z)V} + z^M P(M-1, \xi V)}{\mu + \lambda - (\lambda + b\mu)z}$$

$$\Phi(1, 1, 0, v, 0, 0) = E\left[e^{-vW_\rho}\right] = \frac{\lambda v + b\mu v + a\xi\mu\phi(1, v, 0)}{a\xi\mu + (\lambda + \mu)v}$$

$$\Phi(1, 1, 0, 0, 0, \theta) = E\left[e^{-\theta\tau_\rho}\right] = 1 - \frac{\theta}{\mu + \theta}\left[1 + \frac{b\mu}{\lambda + b\theta} + \frac{a\lambda\mu\phi(1, 0, \theta)}{(\lambda + b\theta)(\lambda + \theta)}\right]$$

Useful Results: CDF of Observed Passage Time, τ_ρ

$$
\begin{aligned}
F_{\tau_\rho}(\vartheta) &= P(\tau_\rho < \vartheta) \\
&= \lambda P(M-1, \xi V) \sum_{j=0}^{M-1} c_j \phi_j(\vartheta) + \lambda e^{-\xi\lambda} \sum_{k=0}^{M-2} \frac{(\xi V)^k}{k!} \sum_{j=0}^{k} d_j \phi_j(\vartheta)
\end{aligned}
$$

where

$$
\begin{aligned}
c_j &= \binom{M-1}{j} (a\lambda)^j b^{M-1-j} \\
d_j &= \binom{k}{j} (a\lambda)^j b^{k-j} \\
\phi_j(\vartheta) &= \frac{1}{\lambda^{j+1}} P(j+1, \lambda\vartheta) - \frac{e^{-\mu\vartheta}}{(\lambda-\mu)^{j+1}} P(j+1, (\lambda-\mu)\vartheta)
\end{aligned}
$$

Useful Results: Means at τ_ρ

$$E[N_\rho] = \frac{\lambda + b\mu}{a\mu} + M - (M-1)Q(M-1, \xi V) + \xi V Q(M-2, \xi V)$$

$$E[W_\rho] = \frac{E[N_\rho]}{\xi} = E[N_\rho]\, E[w_{11}]$$

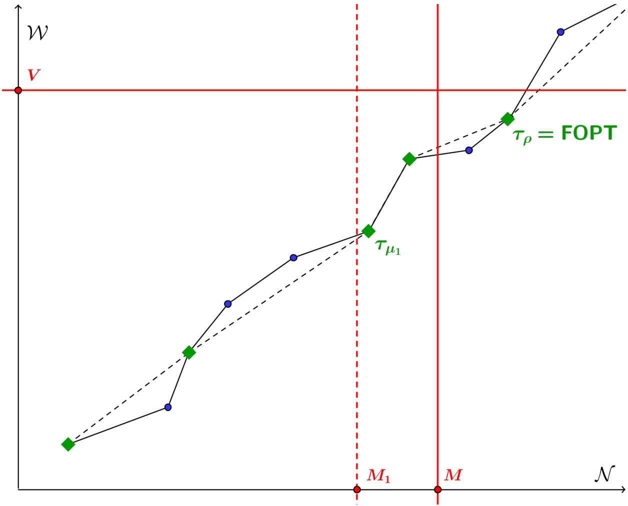
Stochastic Simulation

We simulated 1,000 realizations of the process under each of the some sets of parameters $(\lambda, \mu, a, \xi, M, V)$ and recorded the sample means:

Parameters	$E[N_\rho]$	S. Mean	Error	$E[W_\rho]$	S. Mean	Error
(1)	989.08	988.82	0.26	989.08	990.06	0.98
(2)	990.63	990.39	0.24	990.63	990.27	0.36
(3)	989.28	989.92	0.64	989.28	988.97	0.31
(4)	989.08	989.08	0.00	989.08	989.68	0.31
(5)	503.00	502.73	0.27	1006.00	1005.04	0.96
(6)	1002.00	1001.57	0.43	501.00	500.91	0.09
(7)	803.00	802.68	0.32	803.00	802.67	0.33
(8)	752.00	752.10	0.10	752.00	751.68	0.32
(9)	493.57	493.46	0.11	987.14	986.59	0.55

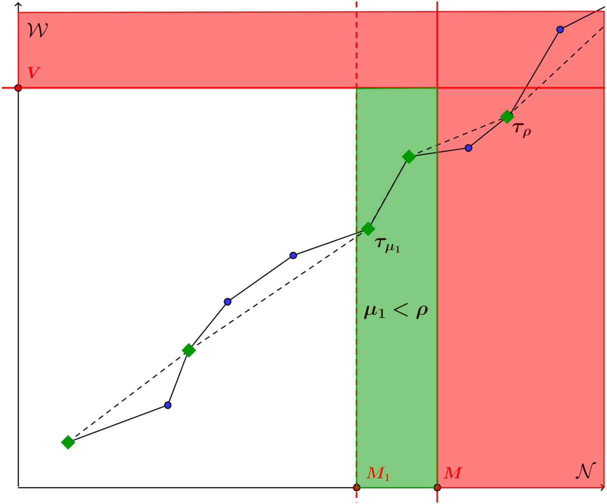
Auxiliary Threshold Model

We introduce another threshold $M_1 < M$ with $\mu_1 = \min\{n : N_n > M_1\}$



Auxiliary Threshold Model

We will be concerned with the confined process where $\mu_1 < \rho$,



Functional of Interest

Our functional of interest will be similar to before, but containing terms associated with μ_1

$$\Phi_{\mu_1<\rho} = \Phi_{\mu_1<\rho}\left(z_0, z, \alpha_0, \alpha, v_0, v, \beta_0, \beta, \theta_0, \theta, h_0, h\right)$$

$$= E\left[\boxed{z_0^{N_{\mu_1-1}} z^{N_{\mu_1}}}\alpha_0^{N_{\rho-1}}\alpha^{N_\rho}\boxed{e^{-v_0 W_{\mu_1-1}-vW_{\mu_1}}}e^{-\beta_0 W_{\rho-1}-\beta W_\rho}\right.$$

$$\left.\times\boxed{e^{-\theta_0\tau_{\mu_1-1}-\theta\tau_{\mu_1}}}e^{-h_0\tau_{\rho-1}-h\tau_\rho}\boxed{\mathbf{1}_{\{\mu_1<\rho\}}}\right]$$

Notice we can find the interesting marginal transform of the time between the two observed threshold crossings,

$$\Phi_{\mu_1>\rho}(1,1,1,1,0,0,0,0,0,-h,0,h) = E\left[e^{-h\left(\tau_\rho-\tau_{\mu_1}\right)}\right]$$

Strategy and Model 1: Constant Observation

We use another operator $\mathcal{D}_{\mu_1}$ adaped to work with the additional threshold and the path to results remains the same

$$\Phi_{\mu_1<\rho} \xrightarrow{\mathcal{D}_{\mu_1}} \Psi_{\mu_1<\rho} \xrightarrow{\text{Assumptions}} \Psi_{\mu_1<\rho} \text{ (explicit) } \xrightarrow{\mathcal{D}_{\mu_1}^{-1}} \Phi_{\mu_1<\rho} \text{ (tractable)}$$

In this model, we have

1. $\Delta_k = c \; a.s.$
2. n_k with **arbitrary** finite distribution $(p_1, p_2, ..., p_m)$
3. $w_{jk} \in [\text{Gamma}(\alpha, \xi)]$

Theorem

$$\Phi_{\mu_1<\rho}(1,z,1,1,0,v,0,0,0,\theta,0,0) = E\left[z^{N_{\mu_1}} e^{-vW_{\mu_1}} e^{-\theta\tau_{\mu_1}} \mathbf{1}_{\{\mu_1<\rho\}}\right]$$

$$= \left\{ \sum_{k=0}^{M_1-1} z^k F_k \sum_{m=0}^{M-1-k} z^m E_m \left(\frac{\xi}{v+\xi}\right)^{\alpha(k+m)} P(\alpha(k+m),(v+\xi)V) \right.$$

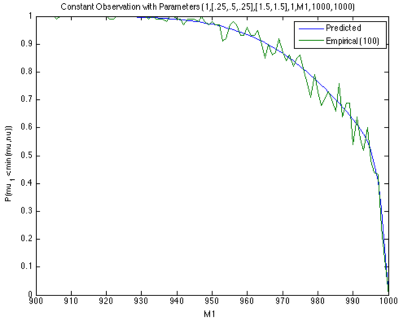
$$\left. - \sum_{k=0}^{M_1-1} z^k \left(\frac{\xi}{v+\xi}\right)^{\alpha k} P(\alpha k,(v+\xi)V) \sum_{n=0}^{k} E_n F_{k-n} \right\} e^{-c(\theta+\lambda)}$$

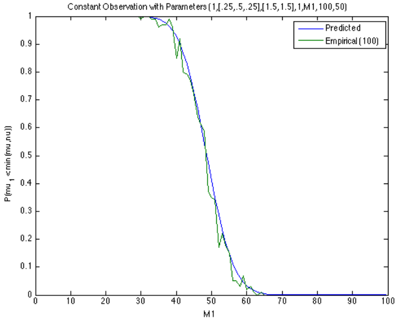
$$F_j = \sum_{r=0}^{\left\lfloor \frac{R-1}{R} j \right\rfloor} (c\lambda)^{j-r} \boxed{Li_{-(j-r)}\left(e^{-c(\theta+\lambda)}\right)} \sum_{\substack{\|\boldsymbol{\beta}\|_1=j \\ [R]\cdot\boldsymbol{\beta}=r+j}} \frac{p_1^{\beta_1}\cdots p_R^{\beta_R}}{\beta_1!\ \cdots\ \beta_R!},$$

$$E_j = \sum_{r=0}^{\left\lfloor \frac{R-1}{R} j \right\rfloor} (c\lambda)^{j-r} \sum_{\substack{\|\boldsymbol{\beta}\|_1=j \\ [R]\cdot\boldsymbol{\beta}=r+j}} \frac{p_1^{\beta_1}\cdots p_R^{\beta_R}}{\beta_1!\ \cdots\ \beta_R!}$$

Results and Simulation

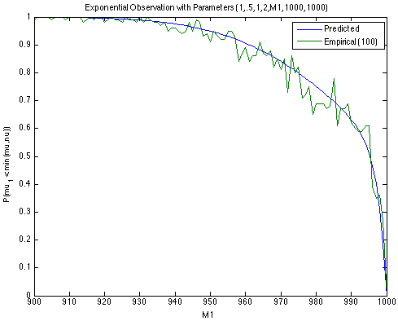
We find $\Phi_{\mu_1<\rho}(1,1,1,1,0,0,0,0,0,0,0,0) = E\left[\mathbf{1}_{\{\mu_1<\rho\}}\right] = P(\mu_1 < \rho)$ and compare to simulated results for a range of M_1 values

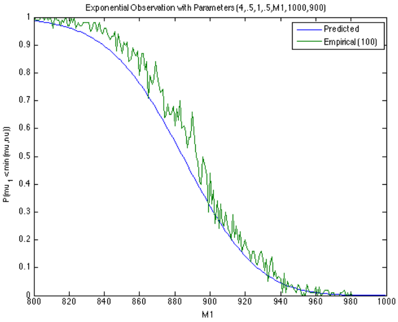




Simulation for an Alternate Model

We did the same under the assumptions $\Delta_1 \in [\text{Exponential}(\mu)]$, $n_1 \in [\text{Geometric}(a)]$, $w_{11} \in [\text{Exponential}(\xi)]$.





Extensions and Future Work

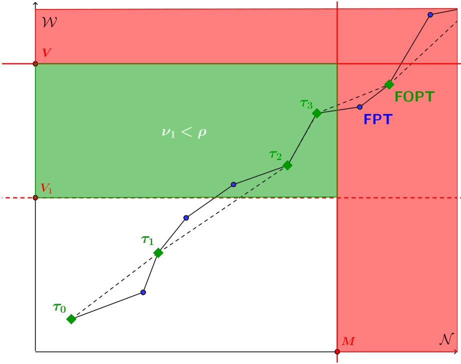


Figure: Continuous Auxiliary Model, $\nu_1 = \min\{n : W_n > V_1\}$

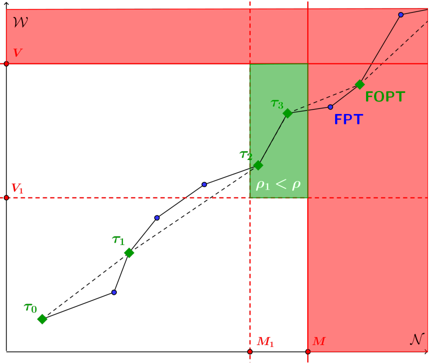
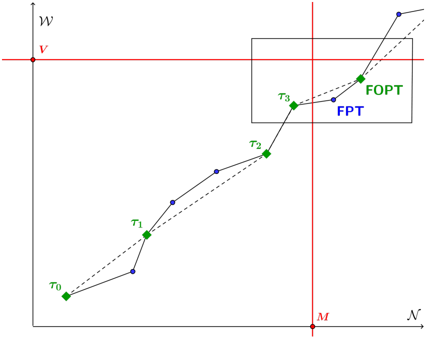


Figure: Dual Auxiliary Model, $\rho_1 = \max\{\mu_1, \nu_1\}$

Extensions and Future Work

- Time-sensitive models
  - We try to “interpolate” the process in random vicinities of the FOPT.



  - We can view n random times of interest
  - Parameters of the process may change at stopping times

Extensions and Future Work

- n-component model
  - n-dimensional process
  - Threshold(s) on each of n components