Time Sensitive Analysis of ISI Processes

Ryan White

FIT Mathematical Sciences Symposium

January 8, 2015

Compound Poisson Process

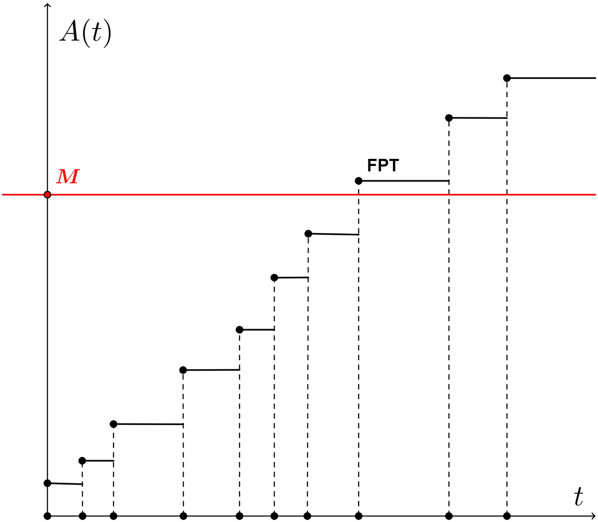
Consider $\mathcal{A}$, a Poisson random measure of rate λ: i.e. for a Poisson point process $t_1 < t_2 < \ldots$ on $\mathbb{R}_{\geq 0}$ and $E \in \mathcal{B}(\mathbb{R}_{\geq 0})$,

- $\mathcal{A}(E) = \sum_{k=1}^{\infty} a_k \varepsilon_{t_k}(E)$ for iid RVs $\{a_k\}$
- ε_{t_k} is the Dirac (point mass) measure,

$$\varepsilon_{t_k}(E) = \begin{cases} 1, & \text{if } t_k \in E \\ 0, & \text{else} \end{cases}$$

- We investigate the compound Poisson process $A(t) = \mathcal{A}([0,t])$ in the vicinity of the first passage time (FPT), $\inf\{t : A(t) > M\}$.

Compound Poisson Process



Delayed Observation

She process is observed upon a delayed renewal process

- $\mathcal{S} = \sum_{n=0}^{\infty} \varepsilon_{\tau_n}$
  - $\Delta_k = \tau_k - \tau_{k-1}$ are *iid* with LST $L(\theta)$, $k \in \mathbb{N}$
  - $\tau_0 = \Delta_0$ is independent of Δ_k, $k \geq 1$

She process of study is actually the **embedded process**

- $Z = \sum_{n=0}^{\infty} \mathcal{A}((\tau_{n-1}, \tau_n])\varepsilon_{\tau_n}$
  - Denote $A_n = Z([0, \tau_n])$, the value of the process at time τ_n

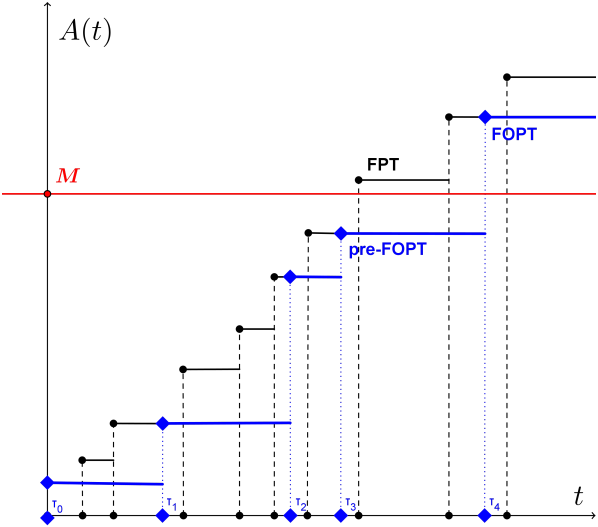
Observed Threshold Crossings

- She **first observed passage time** (FOPT) is τ_ρ, where

$$\rho = \min\{n : A_n > M\}$$

- We also consider the pre-FOPT, $\tau_{\rho-1}$
- Our analysis is restricted by the crudeness of the observations, so we try to approximate information upon the FPT based on probabilistic information upon $\tau_{\rho-1}$ and τ_ρ

Delayed Observation as an Embedded Process



Time Insensitive Results

$$\Phi(v,u,h_0,h) = \mathbb{E}\left[v^{A_{\rho-1}}u^{A_\rho}e^{-h_0\tau_{\rho-1}-h\Delta_\rho}\right]$$

Φ leads to marginal PGFs/LSTs:

- $\Phi(1,u,0,0) = \mathbb{E}\left[u^{A_\rho}\right]$
- $\Phi(1,1,h,h) = \mathbb{E}\left[e^{-h\tau_\rho}\right]$

which lead to moments and distributions of components of the process upon $\tau_{\rho-1}$ and τ_ρ:

- $\mathbb{E}[A_\rho]$, $\mathbb{P}\{A_\rho = n\}$
- $\mathbb{E}[\tau_\rho]$, $\mathbb{P}\{\tau_\rho > t\}$

Time Sensitive Results

$$\Phi_1(t,v,u,z,h_0,h) = \mathbb{E}\left[v^{A_{\rho-1}} u^{A_\rho} z^{A(t)} e^{-h_0\tau_{\rho-1}-h\Delta_\rho} \mathbf{1}_{\{t<\tau_{\rho-1}\}}\right]$$
$$\Phi_2(t,v,u,z,h_0,h) = \mathbb{E}\left[v^{A_{\rho-1}} u^{A_\rho} z^{A(t)} e^{-h_0\tau_{\rho-1}-h\Delta_\rho} \mathbf{1}_{\{\tau_{\rho-1}\le t<\tau_\rho\}}\right]$$
$$\Phi(t,v,u,z,h_0,h) = \mathbb{E}\left[v^{A_{\rho-1}} u^{A_\rho} z^{A(t)} e^{-h_0\tau_{\rho-1}-h\Delta_\rho} \mathbf{1}_{\{t<\tau_\rho\}}\right]$$

New capabilities:

- Joint results: $\mathbb{E}\left[u^{A_\rho}\mathbf{1}_{\{t<\tau_\rho\}}\right]$, $\mathbb{P}\left\{A_\rho = n, \tau_\rho > t\right\}$
- Conditional probabilities: $\mathbb{P}\{A_\rho = n | \tau_\rho > t\} = \frac{\mathbb{P}\{A_\rho=n,\tau_\rho>t\}}{\mathbb{P}\{\tau_\rho>t\}}$

ISI Processes

Consider a stochastic process $A(t)$ on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ such that

- $A(t)$ has independent increments:
  - For $0 < t_1 < ... < t_k$, the increments $A(t_1) - A(0), A(t_2) - A(t_1), ..., A(t_k) - A(t_{k-1})$ are independent.
  - i.e., increments on non-overlapping time intervals are independent
- $A(t)$ has stationary increments:
  - For $0 \leq s < t$, the distribution of $A(t) - A(s)$ depends only on $t - s$
- Lévy processes are ISI (e.g. Poisson processes, Wiener processes)

Theorem 1

Let S and Δ be RVs independent of each other and of $\mathcal{F}_t$, then on the trace σ-algebra $\mathcal{F} \cap \{t < S\}$, the functional

$$F_1(t, v, u, z, h_0, h) = \mathbb{E}\left[v^{A(S)} u^{A(S+\Delta)} z^{A(t)} e^{-h_0 S - h\Delta} \mathbf{1}_{\{t<S\}}\right]$$

satisfies

$$F_1^*(\theta, v, u, z, h_0, h) = \mathbb{E}\left[e^{-h_0 S} \psi(uvz, uv, S)\right] \mathbb{E}\left[e^{-h\Delta} \varphi(u, \Delta)\right]$$

where

- $f^*(t) = \mathcal{L}_t\{f\}(\theta)$
- $\varphi(a, s) = \mathbb{E}\left[a^{A(s)}\right]$
- $\psi(b, c, \delta) = \left(e^{-\theta(\cdot)} \varphi(b, \cdot)\right) * \varphi(c, \cdot)(\delta) = \int_0^\delta e^{-\theta t} \varphi(b, t) \varphi(c, \delta - t)\, dt$

Theorem 2

Let S and Δ be RVs independent of each other and of $\mathcal{F}_t$, then on the trace σ-algebra $\mathcal{F} \cap \{S \leq t < S + \Delta\}$, the functional

$$F_2(t, v, u, z, h_0, h) = \mathbb{E}\left[v^{A(S)} u^{A(S+\Delta)} z^{A(t)} e^{-h_0 S - h(S+\Delta)} \mathbf{1}_{\{S \leq t < S+\Delta\}}\right]$$

satisfies

$$F_2^*(\theta, v, u, z, h_0, h) = \mathbb{E}\left[e^{-h_0+\theta} \varphi(uvz, S)\right] \mathbb{E}\left[e^{-h\Delta} \psi(uz, u, \Delta)\right]$$

Theorem 2 Proof

Since

$$A(t) = A(t) - A(S) + A(S)$$
$$A(S+\Delta) = A(S+\Delta) - A(t) + A(t) - A(S) + A(S),$$

$\mathcal{F}_t$ is independent of the σ-algebra $\sigma(S,\Delta)$ and A is an ISI process,

$$\begin{aligned}
F_2 &= \mathbb{E}\left[e^{-h_0 S - h\Delta}\mathbf{1}_{\{S\le t<S+\Delta\}}\mathbb{E}\left[v^{A(S)}u^{A(S+\Delta)}z^{A(t)}|\sigma(S,\Delta)\right]\right] \\
&= \mathbb{E}\left[e^{-h_0 S - h\Delta}\mathbf{1}_{\{S\le t<S+\Delta\}}\mathbb{E}\left[(uvz)^{A(S)}(uz)^{A(t)-A(S)}u^{A(S+\Delta)-A(t)}|\sigma(S,\Delta)\right]\right] \\
&= \mathbb{E}\left[e^{-h_0 S - h\Delta}\mathbf{1}_{\{S\le t<S+\Delta\}}\varphi(uvz,S)\varphi(uz,t-S)\varphi(u,S+\Delta-t)\right]
\end{aligned}$$

Theorem 2 Proof (cont.)

By Fubini's Theorem and the independence of S and Δ,

$$F_2^* = \int_{t\geq 0} e^{-\theta t} \int_{(s,\delta)\in\mathbb{R}^2_{\geq 0}} e^{-h_0 s - h\delta}\varphi(uvz,s)\varphi(uz,t-s)\varphi(u,s+\delta-t) \times \mathbf{1}_{\{s\leq t<s+\delta\}}\, d\mathbb{P}_{S\otimes\Delta}(s,\delta)\, dt \tag{1}$$

$$= \int_{\delta\geq 0} e^{-h\delta} \int_{t=s}^{s+\delta} e^{\theta(t-s)}\varphi(uz,t-s)\varphi(u,s+\delta-t)\, dt\, d\mathbb{P}_\Delta(\delta) \times \int_{s\geq 0} e^{-(h_0+\theta)s}\varphi(uvz,s)\, d\mathbb{P}_S(s) \tag{2}$$

Theorem 2 Proof (cont.)

By the translation invariance of the Borel-Lebesgue measure and setting $w = t - s$,

$$F_2^* = \int\limits_{\delta \geq 0} e^{-h\delta} \left(\int\limits_{w=0}^{\delta} e^{-\theta w} \varphi(uz, w) \varphi(u, \delta - w) \, dw \right) d\mathbb{P}_\Delta(\delta) \times \mathbb{E}\left[e^{(h_0+\theta)S} \varphi(uvz, S) \right] \tag{3}$$

$$= \int\limits_{\delta \geq 0} e^{-h\delta} \psi(uz, u, \delta) \, d\mathbb{P}_\Delta(\delta) \, \mathbb{E}\left[e^{(h_0+\theta)S} \varphi(uvz, S) \right] \tag{4}$$

$$= \mathbb{E}\left[e^{-h\Delta} \psi(uz, u, \Delta) \right] \mathbb{E}\left[e^{(h_0+\theta)S} \varphi(uvz, S) \right] \tag{5}$$

Lemma 3

- If $A(t)$ is a compound Poisson process, then

$$
\begin{aligned}
&F_2^*(\theta, t, v, u, z, h_0, h) \\
&= L_S\left(\theta + h_0 + \lambda - \lambda g(uvz)\right) \\
&\quad \times \frac{L_\Delta\left(h + \lambda - \lambda g(u)\right) - L_\Delta\left(\theta + h + \lambda - \lambda g(uz)\right)}{\theta + \lambda g(u) - \lambda g(uz)}
\end{aligned}
$$

  where

  - $L_X(\theta) = \mathbb{E}\left[e^{-\theta X}\right]$ (the LST of RV X)
  - $g(z) = \mathbb{E}\left[z^{a_k}\right]$ (the PGF of a_k's)

Returning to the Original Goal

- One goal was the functional

$$\Phi_2(t, v, u, z, h_0, h) = \mathbb{E}\left[v^{A_{\rho-1}} u^{A_\rho} z^{A(t)} e^{-h_0\tau_{\rho-1} - h\Delta_\rho} \mathbf{1}_{\{\tau_{\rho-1} \le t < \tau_\rho\}}\right]$$

  for a compound Poisson process $A(t)$

- $\tau_{\rho-1}$ and $\Delta_\rho = \tau_\rho - \tau_{\rho-1}$ are not independent, so it does not immediately apply

- However, τ_j and Δ_j are independent for fixed $j \in \mathbb{Z}_{\geq 0}$

Finding Φ_2^*

- We find Φ_2 by summing it over fixed j's, i.e., notice

$$v^{A_{\rho-1}} u^{A_\rho} z^{A(t)} e^{-h_0\tau_{\rho-1} - h\Delta_\rho} \mathbf{1}_{\{\tau_{\rho-1} \le t < \tau_\rho\}} = \sum_{j \ge 0} v^{A_{j-1}} u^{A_j} z^{A(t)} e^{-h_0\tau_{j-1} - h\Delta_j} \mathbf{1}_{\{\rho = j\}} \mathbf{1}_{\{\tau_{\rho-1} \le t < \tau_\rho\}}$$

- Let

$$F_{2,j} = \mathbb{E}\left[v^{A_{j-1}} u^{A_j} z^{A(t)} e^{-h_0\tau_{j-1} - h\Delta_j} \mathbf{1}_{\{\tau_{j-1} \le t < \tau_j\}} \right]$$

- Then with an operator similar to a z-transform $\mathcal{D}_p\{\cdot\}(y)$ with inverse $\mathcal{D}_y^{M-1}$ (details omitted),

$$\Phi_2^* = \mathcal{D}_y^{M-1} \Big\{ F_{2,0}^*(t,1,u,z,0,h) - F_{2,0}^*(t,1,uy,z,0,h) + \sum_{j>0} F_{2,j}^*(t,vy,u,z,h_0,h) - F_{2,j}^*(t,v,uy,z,h_0,h) \Big\}$$

Theorem 4

- The joint functional $\Phi_2^*(\theta)$ of $A(t)$ on the interval $[\tau_{\rho-1}, \tau_\rho)$ satisfies

$$\Phi_2^* = \mathcal{D}_y^{M-1}\left\{\Gamma_0 + \Gamma B_3\right\}$$

where

- $\gamma_0(z,\theta) = \mathbb{E}\left[z^{A_0}e^{-\theta\Delta_0}\right]$
- $\gamma(z,\theta) = \mathbb{E}\left[z^{A_1-A_0}e^{-\theta\Delta_1}\right]$
- $\Gamma_0 = \frac{\gamma_0(u,h)-\gamma_0(uz,\theta+h)}{\theta+\lambda g(u)-\lambda g(uz)} - \frac{\gamma_0(uy,h)-\gamma_0(uzy,\theta+h)}{\theta+\lambda g(uy)-\lambda g(uzy)}$
- $\Gamma = \frac{\gamma(u,h)-\gamma(uz,\theta+h)}{\theta+\lambda g(u)-\lambda g(uz)} - \frac{\gamma(uy,h)-\gamma(uzy,\theta+h)}{\theta+\lambda g(uy)-\lambda g(uzy)}$
- $B_3 = \frac{\gamma_0(vuzy,\theta+h_0)}{1-\gamma(vuzy,\theta+h_0)}$

Results for a Special Case

- Suppose

  1. a_k are geometrically distributed with parameter a: $g(z) = \frac{az}{1-bz}$
  2. Δ_k are exponentially distributed with parameter μ: $L_{\Delta_1}(z) = \frac{\mu}{\mu+z}$
  3. $\tau_0 = 0$: $\gamma_0(\cdot, \cdot) = 1$

Results for a Special Case (cont.)

- A demonstration of finding useful results follows

  1. Use g and L to simplify Theorem 4 (the $\{t < \tau_{\rho-1}\}$ version) and take the inverse $\mathcal{D}_y^{M-1}$ to get

  $$\Phi_1^*(\theta, 1, u, 1, 0, 0) = \mathcal{L}_t \left\{ \mathbb{E}\left[u^{A_\rho} \mathbf{1}_{\{t<\tau_{\rho-1}\}} \right] \right\} (\theta)$$

  2. Find the inverse Laplace transform to find

  $$\Phi_1(\theta, 1, u, 1, 0, 0) = \mathbb{E}\left[u^{A_\rho} \mathbf{1}_{\{t<\tau_{\rho-1}\}} \right]$$

  3. This a restricted PGF, so apply $\frac{1}{n!} \lim_{u \to 0} \frac{\partial^n}{\partial u^n}(\cdot)$ to find

  $$\mathbb{P}\{A_\rho = n, \tau_{\rho-1} > t\}$$

- We followed this process in [3] to find a long, but analytically and numerically tractable formula.

Extensions and Future Research

- d-Dimensional Process
  - For $\mathbf{\Delta} = (S_1, S_2 - S_1) = (\Delta_1, \Delta_2)$,

$$\Phi_1(t, \mathbf{v}_1, \mathbf{v}_2, \mathbf{w}, \mathbf{x}) = \mathbb{E}\left[e^{-\mathbf{v}_1 \cdot A(S_1) - \mathbf{v}_2 \cdot A(S_2) - \mathbf{w} \cdot A(t) - \mathbf{x} \cdot \mathbf{\Delta}} \mathbf{1}_{\{t < S_1\}}\right]$$

$$\Phi_2(t, \mathbf{v}_1, \mathbf{v}_2, \mathbf{w}, \mathbf{x}) = \mathbb{E}\left[e^{-\mathbf{v}_1 \cdot A(S_1) - \mathbf{v}_2 \cdot A(S_2) - \mathbf{w} \cdot A(t) - \mathbf{x} \cdot \mathbf{\Delta}} \mathbf{1}_{\{S_1 \leq t < S_1 + \Delta_1\}}\right]$$

- m Times of Interest in d Dimensions
  - For $S_0 = 0$, $1 \leq n \leq m$, and $\mathbf{\Delta} = (\Delta_1, ..., \Delta_m)$,

$$\Phi_n(t, \mathbf{v}_1, ..., \mathbf{v}_m, \mathbf{w}, \mathbf{x}) = \mathbb{E}\left[e^{-\sum_{j=1}^{m} \mathbf{v}_j \cdot A(S_j) - \mathbf{w} \cdot A(t) - \mathbf{x} \cdot \mathbf{\Delta}} \mathbf{1}_{\{S_{n-1} \leq t < S_{n-1} + \Delta_n\}}\right]$$

References

1 P. Billingsley. *Probability and Measure*, Third Edition (1996)

2 J. H. Dshalalow and R. White. On Strategic Defense in Stochastic Networks. *Stochastic Analysis and Applications*, 32:3 (2014), 365-396

3 J. H. Dshalalow and R. White. Time Sensitive Analysis of Independent and Stationary Increment Processes. [Submitted]