On Exits of Oscillating Random Walks Under Delayed Observation

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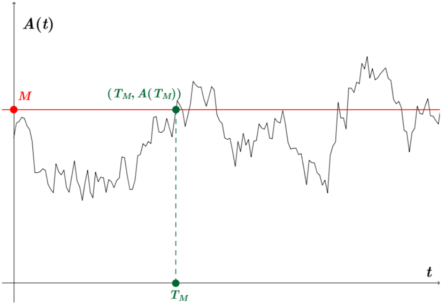
A Standard Problem

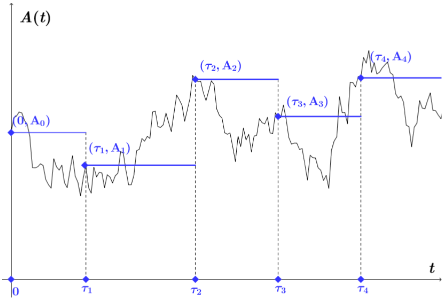
- Let $A(t)$ be a Levy process on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})$
- We consider the **first exit** of $A(t)$ from $(-\infty, M)$, i.e. a level crossing of M
- The **first exit time** is $T_M = \inf\{t : A(t) > M\}$



Delayed Observation

- $A(t)$, the real-time process, is assumed to be inaccessible to the observer – the position of A can be observed only upon a renewal process $\{\tau_j\}$

- The **observed process** is the piecewise-constant subsequence $\{A(\tau_j)\} := \{A_j\}$.

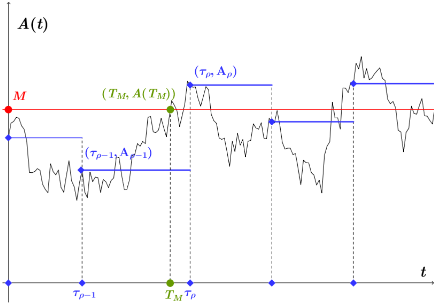


Realtime Process with **Observed Process**

Complications of Delayed Observation

- The first exit time $T_M \notin \{\tau_j\}$ *a.s.*, so the value of $A(T_M)$ is *a.s.* inaccessible to the observer

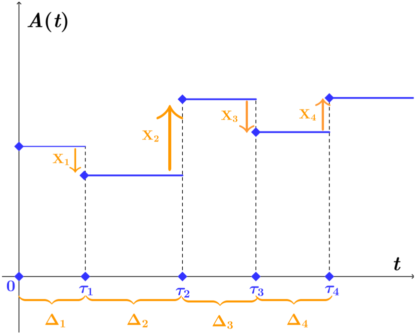
- Consider the **post-exit time** τ_ρ for $\rho = \inf\{n : A(\tau_n) > M\}$ and **pre-exit time** $\tau_{\rho-1}$



Realtime Process with **Observed Process**, **Threshold**, and **First Exit Time**

Increments of the Observed Process

- Denote $(\Delta_j, X_j) = (\tau_j - \tau_{j-1}, A_j - A_{j-1})$



The **Observed Process** and **Increments**

- Assume (Δ_j, X_j) are **jointly *i.i.d.*** random vectors with **dependent** components
  - $|X_j|$ and Δ_j are likely to be positively correlated in applications

Applications of Continuous Time Random Walks (CTRWs)

- Modeling anomalous diffusion in physics
  [Stanislavsky and Weron, 2010, Metzler and Klafter, 2007]
- Reliability of large-scale networks under attack and/or benign node failure
  [Dshalalow and White, 2013, 2014, White, 2015]
- Stochastic games
  [Dshalalow and Treerattrakoon, 2010]
- Geophysical motion of particles on planetary surfaces
  [Schumer et al., 2009]
- Financial forensics and modeling
  [Scalas, 2004, Jurlewicz et al., 2009, Dshalalow and Liew, 2006]

Known Results for Monotone CTRW Exiting $(-\infty, M]$

- Joint Laplace-Stieltjes transforms:

$$\mathbb{E}\left[e^{-\alpha_0 A_{\rho-1}-\alpha A_\rho - h_0\tau_{\rho-1}-h\tau_\rho}\right]$$

  - Marginal Laplace-Stieltjes transforms, e.g. $\mathbb{E}\left[e^{-\alpha A_{\tau_\rho}}\right]$
  - Marginal distributions and moments (tractable in special cases)
- **Joint** distributions, e.g. $\mathbb{P}\{A_\rho < s, \tau_\rho < t\}$
- Similar results in 2-3 spatial dimension CTRW exiting rectangles
- Similar results for monotone “streaks” within oscillating CTRW

An Operational Calculus Approach to Monotone CTRW

1. Consider the family $\{\rho(q) = \inf\{n : A_n > q\}$ and partition the sample space

$$\mathbb{E}\left[e^{-\alpha_0 A_{\rho(q)-1} - \alpha A_{\rho(q)} - h_0 \tau_{\rho(q)-1} - h\tau_{\rho(q)}}\right]$$

$$= \sum_{j=0}^{\infty} \mathbb{E}\left[e^{-\alpha_0 A_{j-1} - \alpha A_j - h_0 \tau_{j-1} - h\tau_j} \underbrace{\mathbf{1}_{\{\rho(q)=j\}}}_{\text{The only } q\text{-dependent term}}\right]$$

2. Apply the Laplace-Carson transform $\left(x \int_{q\geq 0} e^{-xq} f(q)\, dq\right)$

$$\sum_{j=0}^{\infty} \mathbb{E}\left[e^{-\alpha_0 A_{j-1} - \alpha A_j - h_0 \tau_{j-1} - h\tau_j} \left(e^{-xA_{j-1}} - e^{-xA_j}\right)\right]$$

3. Exploit independent and stationary increments

$$\sum_{j=0}^{\infty} \underbrace{\mathbb{E}\left[e^{-(\alpha_0+\alpha+x)X_1 - (h_0+h)\Delta_1}\right]^{j-1}}_{\text{corresponds to } [0, \tau_{\rho-1}]} \underbrace{\left(\mathbb{E}\left[e^{-\alpha X_1 - h\Delta_1}\right] - \mathbb{E}\left[e^{-(\alpha+x)X_1 - h\Delta_1}\right]\right)}_{\text{correpsonds to } (\tau_{\rho-1}, \tau_\rho]}$$

4. Sum as a geometric series
5. Invert the Laplace-Carson under special cases with $q \to M$

Monotone CTRW Result

- It is known

$$\Phi(\alpha_0,\alpha,h_0,h) = \mathbb{E}\left[e^{-\alpha_0 A_{\rho-1}-\alpha A_\rho - h_0\tau_{\rho-1}-h\tau_\rho}\right]$$
$$= \mathcal{L}\mathcal{C}_x^{-1}\left(\frac{\gamma(\alpha,h)-\gamma(\alpha+x,h)}{1-\gamma(\alpha_0+\alpha+x,h_0+h)}\right)(M),$$

  where

$$\gamma(\alpha,h) = \underbrace{\mathbb{E}\left[e^{-\alpha X_1 - h\Delta_1}\right]}_{\text{joint LST of an increment}}$$

- Inversions for many special cases are analytically or numerically tractable
- Similar approaches work in more dimensions

The Monotone Approach Fails

- The approach above does not work with general CTRW **(convergence may fail)**
- Consider the monotone increasing "streaks" broken up by "drops" in the CTRW, e.g.

$X_1 \geq 0$	$X_2 < 0$	$X_3 \geq 0$	$X_4 \geq 0$	$X_5 < 0$	$X_6 \geq 0$	...	$X_\rho \geq 0$
$\nearrow$	$\searrow$	$\nearrow$	$\nearrow$	$\searrow$	$\nearrow$	...	$\nearrow$
monotone $\uparrow$	**DROP**	monotone $\uparrow$		**DROP**	monotone $\uparrow$		

- **The idea**: exploit the monotone approach on the increasing "streaks"

Partitioning the Space

- Summing over every possible sequence of increasing "streaks" is unrealistic
- Can we partition the sample space usefully?
  - Restrict interest to the event $\mathcal{E} = \{A_k \text{ eventually exits}\}$
  - Partition into the events $\mathcal{E}_n = \{A_k \text{ exits between the } (n-1)\text{th and } n\text{th drop}\}$

$$
\begin{aligned}
\Phi_{\mathcal{E}}(\alpha_0, \alpha, h_0, h) &:= \mathbb{E}\left[e^{i\alpha_0 A_{\rho-1} + i\alpha A_\rho + ih_0\tau_{\rho-1} + ih\tau_\rho}\mathbf{1}_{\mathcal{E}}\right] \\
&= \sum_{n=0}^{\infty} \mathbb{E}\left[e^{i\alpha_0 A_{\rho-1} + i\alpha A_\rho + ih_0\tau_{\rho-1} + ih\tau_\rho}\mathbf{1}_{\mathcal{E}_n}\right] \\
&=: \sum_{k=0}^{\infty} \Phi_{\mathcal{E}_n}(\alpha_0, \alpha, h_0, h)
\end{aligned}
$$

Calculating $\Phi_{\mathcal{E}_n}$ (I)

- **"Drop" indices**

$$\mu_1 = \inf\{m : X_m < 0\} \qquad \text{(the first "drop")}$$

$$\mu_n = \inf\{m : X_m < 0, m > \mu_{n-1}\} \qquad \text{(the } n\text{th "drop")}$$

- For $n > 2$, sum for all possible $\mu_{n-1} < \rho < \mu_n$

$$\Phi_{\mathcal{E}_n}(\alpha_0, \alpha, h_0, h)$$

$$= \mathbb{E}\left[e^{i\alpha_0 A_{\rho-1} + i\alpha A_\rho + ih_0\tau_{\rho-1} + ih\tau_\rho} \mathbf{1}_{\{\mu_{n-1}<\rho<\mu_n\}}\right]$$

$$= \sum_{j=n-1}^{\infty} \sum_{k=j+1}^{\infty} \sum_{l=k+1}^{\infty} \mathbb{E}\left[e^{i\alpha_0 A_{k-1} + i\alpha A_k + ih_0\tau_{k-1} + ih\tau_k} \mathbf{1}_{\{\mu_{n-1}=j, \rho=k, \mu_n=l\}}\right]$$

Side-Stepping a Combinatorial Nightmare

- The indicator $\mathbf{1}_{\{\mu_{n-1}=j,\rho=k,\mu_n=l\}}$ allows for $n-2$ “drop” indices below j
  - i.e. $\mu_1 < \mu_2 < \cdots < \mu_{n-2} < j$
  - Each has the same **number** of “drops” and positive jumps, i.e. $\binom{j-1}{n-2}$ **equal** terms
- Apply **Fourier**-Carson transform and exploit independent and stationary increments:

$$\mathbb{E}\left[e^{i\alpha_0 A_{k-1}+i\alpha A_k+ih_0\tau_{k-1}+ih\tau_k}\mathbf{1}_{\{\mu_{n-1}=j,\rho=k,\mu_n=l\}}\right]^*$$
$$=\binom{j-1}{n-2}\left(\gamma^+\right)^{j-(n-1)}\left(\gamma^-\right)^{n-1}\left(\sigma^1-\sigma\right)\left(\gamma^+\right)^{k-j-1}P\{X_1\geq 0\}^{l-k-1}P\{X_1<0\}$$

- $\delta^+(\alpha,h)=\underbrace{\mathbb{E}\left[e^{i\alpha X_1+ih\Delta_1}\mathbf{1}_{\{X_1\geq 0\}}\right]}_{\text{joint CF of nonnegative increment}}$
- $\delta^-(\alpha,h)=\underbrace{\mathbb{E}\left[e^{i\alpha X_1+ih\Delta_1}\mathbf{1}_{\{X_1< 0\}}\right]}_{\text{joint CF of negative increment}}$
- $\gamma^+=\delta^+(\alpha_0+\alpha+x,h_0+h)$ $\quad\gamma^-=\delta^-(\alpha_0+\alpha+x,h_0+h)$
- $\sigma^1=\delta^+(\alpha,h)$ $\quad\sigma=\delta^+(\alpha+x,h)$

Calculating $\Phi_{\mathcal{E}_n}$ (II)

- Simple calculations show

$$\Phi^*_{\mathcal{E}_n}(\alpha_0, \alpha, h_0, h)$$

$$= p^- \left(\sigma^1 - \sigma\right) \left(\gamma^-\right)^{n-1} \sum_{j=n-1}^{\infty} \binom{j-1}{n-2} \left(\gamma^+\right)^{j-(n-1)} \sum_{k=j+1}^{\infty} \left(\gamma^+\right)^{k-j-1} \sum_{l=k+1}^{\infty} \left(p^+\right)^{l-k-1}$$

$$= \frac{\sigma^1 - \sigma}{1 - \gamma^+} \left(\frac{\gamma^-}{1 - \gamma^+} \right)^{n-1}$$

Calculating $\Phi_{\mathcal{E}}$

- Simply use the geometric series,

$$\Phi^*_{\mathcal{E}}(\alpha_0, \alpha, h_0, h) = \sum_{k=0}^{\infty} \Phi^*_{\mathcal{E}_n}(\alpha_0, \alpha, h_0, h)$$
$$= \frac{\sigma^1 - \sigma}{1 - \gamma^+ - \gamma^-}$$
$$= \frac{\sigma^1 - \sigma}{1 - \gamma}$$

where $\gamma = \mathbb{E}\left[e^{i(\alpha_0+\alpha+x)X_1+i(h_0+h)\Delta_1}\right]$ (joint CF of increment)

- Invert the Fourier-Carson transform

$$\Phi_{\mathcal{E}}(\alpha_0, \alpha, h_0, h) = \mathcal{FC}_x^{-1}\left(\frac{\sigma^1 - \sigma}{1 - \gamma}\right)(M)$$

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