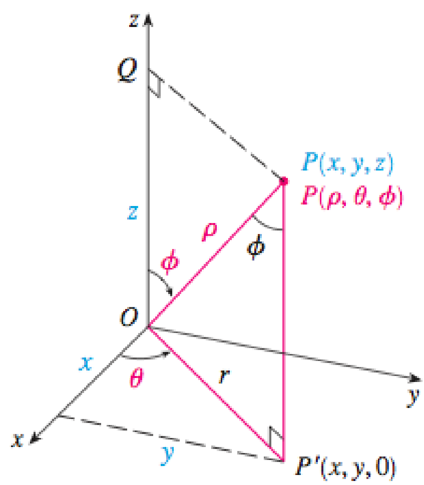
§15.8 Triple Integrals in Cylindrical Coordinates

- Cylindrical coordinates describe points in 3D space by (r, θ, z) where (r, θ) are the polar coordinates of (x, y) and z remains as in cartesian coordinates.
  - That is, $x = r\cos\theta$, $y = r\sin\theta$, and $z = z$.
- $\iiint_E f(x,y,z)dV = \int_\alpha^\beta \int_{h_1(\theta)}^{h_2(\theta)} \int_{u_1(r\cos\theta, r\sin\theta)}^{u_2(r\cos\theta, r\sin\theta)} f(r\cos\theta, r\sin\theta, z) r\, dz\, dr\, d\theta$
  - where $E = \{(x,y,z) : (x,y) \in D, u_1(x,y) \le z \le u_2(x,y)\}$ and $D = \{(r,\theta) \,|\, \alpha \le \theta \le \beta, h_1(\theta) \le r \le h_2(\theta)\}$

§15.9 Triple Integrals in Spherical Coordinates



- $x = \rho\sin\phi\cos\theta$
- $y = \rho\sin\phi\sin\theta$
- $z = \rho\cos\phi$
- In some fields, spherical coordinates are defined differently.

- $\iiint_E f(x,y,z)dV = \int_c^d \int_\alpha^\beta \int_a^b f(\rho\sin\phi\cos\theta, \rho\sin\phi\sin\theta, \rho\cos\phi)\rho^2\sin\phi\, d\rho\, d\theta\, d\phi$ where $E = \{(\rho,\theta,\phi) : a \le \rho \le b, \alpha \le \theta \le \beta, c \le \phi \le d\}$
  - That is, we convert to spherical coordinates and multiply by $\rho^2\sin\phi$ before evaluating the integral.

Problems

Example 1, §15.3: Find the volume of the solid under the surface $z = y$ and above the triangle with vertices (0,2), (3,2), and (1,1).

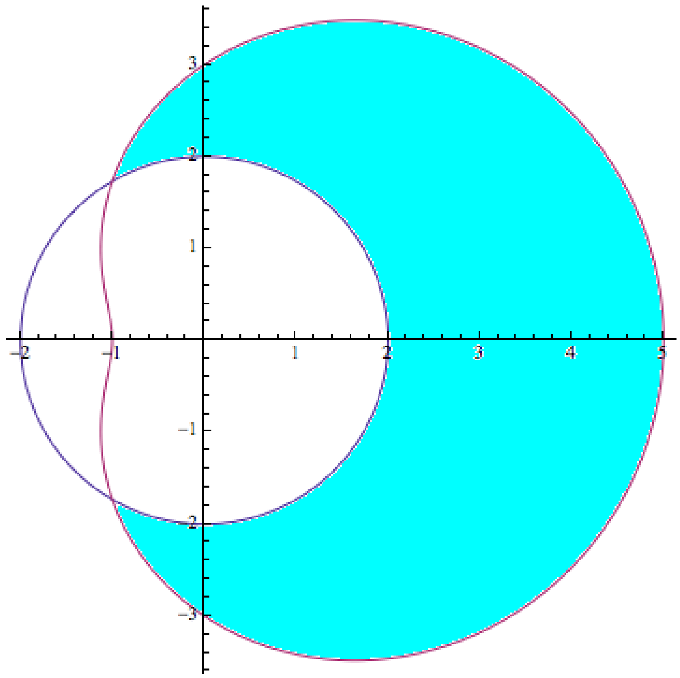
First, draw the triangle – the upper side is just y=2, and so we see $1 \leq y \leq 2$. The bounds on x are a little more complicated, as we need the horizontal distance between the other two sides of the triangle given x, so we need to find the equations of these sides, which are straight lines.

The line connecting (0,2) and (1,1) has slope -1 and y-intercept 2, so the line is $y = -x + 2$, but we want it to be in terms of x, so solving for x, we have $x = 2 - y$, which will be the lower (left) bound on x. The other line is between (1,1) and (3,2), so the slope is 1/2, so an equation of the line (point-slope form) is $(y - 1) = (1/2)(x - 1)$, and we solve for x to find $2y - 1 = x$ as the upper bound for x, which allows us to calculate the integral:

$$\begin{aligned} V &= \int_1^2 \int_{2-y}^{2y-1} y\,dx\,dy = \int_1^2 y \int_{2-y}^{2y-1} dx\,dy \\ &= \int_1^2 y[2y - 1 - 2 + y]\,dx = \int_1^2 3y^2 - 3y\,dy \\ &= 3\left[\frac{y^3}{3} - \frac{y^2}{2}\right]_1^2 = 3\left[\frac{8}{3} - 2 - \frac{1}{3} + \frac{1}{2}\right] = \frac{5}{2} \end{aligned}$$

Example 2, §15.4: Find the area of the region inside $r = 3 + 2\cos\theta$ and outside a circle of radius 2 centered at the origin.

Solution. First, note the formula for such a circle in polar coordinates is simply $r = 2$. Since $1 \leq \sin\theta \leq 1$, the first figure's radius must vary between -1 and 5. It should be clear how to roughly draw such a figure – it doesn't have to be great, the reason for drawing is really just to show us roughly what the enclosed region looks like so that we understand where to integrate once we find the points of intersection.



So, we see that everywhere, the bound on r will be $2 \leq r \leq 3 + 2\cos\theta$, and the region has symmetry about the x-axis, so we can find the area of the top half of the shaded region and multiply by 2. As such, the upper bound on θ will be the point of intersection in the second quadrant, which we will find now

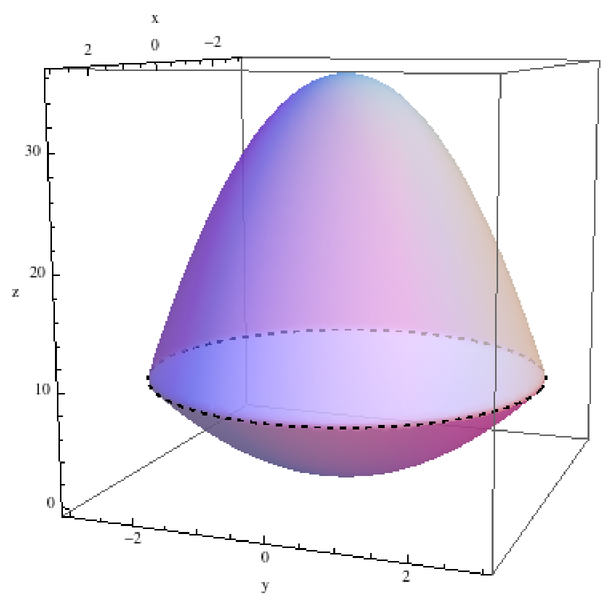
$$\begin{aligned} 2 &= 3 + 2\cos\theta \\ -\frac{1}{2} &= \cos\theta \\ \theta &= \frac{2\pi}{3}, \frac{4\pi}{3} \end{aligned}$$

The $\frac{2\pi}{3}$ is the one in the second quadrant, so we can now do the integral:

$$A = 2\int_0^{2\pi/3}\int_2^{3+2\cos\theta} r\,dr\,d\theta = 2\int_0^{2\pi/3}\left[\frac{r^2}{2}\right]_2^{3+2\cos\theta}\,d\theta$$
$$= \int_0^{2\pi/3} 9 + 12\cos\theta + 4\cos^2\theta - 4\,d\theta$$
$$= \int_0^{2\pi/3} 7 + 12\cos\theta + 2\cos 2\theta\,d\theta$$
$$= [7\theta + 12\sin\theta + \sin 2\theta]_0^{2\pi/3}$$
$$= \frac{14\pi}{3} + 6\sqrt{3} - \frac{\sqrt{3}}{2} = \boldsymbol{\frac{14\pi}{3} + \frac{11\sqrt{3}}{2}}$$

Example 3, # 25 (a), §15.8: Find the volume of the region E bounded by the paraboloids $z = x^2 + y^2$ and $z = 36 - 3x^2 - 3y^2$.

Solution. Cylindrical coordinates will work easily here since we already have simple bounds on z. We should see this region is the intersection of two rounded cones, one pointing up and the other pointing down, to get a shape like this:



The dotted line marks the intersection of the two paraboloids because this is the widest portion and so it will be the portion projected onto the xy-plane, which we will need in order to bound r and θ. We can solve for the intersection by setting them equal to one another:

$$\begin{aligned} x^2 + y^2 &= 36 - 3x^2 - 3y^2 \\ 4x^2 + 4y^2 &= 36 \\ x^2 + y^2 &= 9 \end{aligned}$$

So the projection on the xy-plane is simply a circle centered at $(0, 0)$ of radius 3. If we are to bound this region by polar coordinates, we have $0 \leq r \leq 3$ and $0 \leq \theta \leq 2\pi$.

The bound on z should be the distance between the two paraboloids for a given (r, θ), and the $z = 36 - 3x^2 - 3y^2$ is on top, so we have

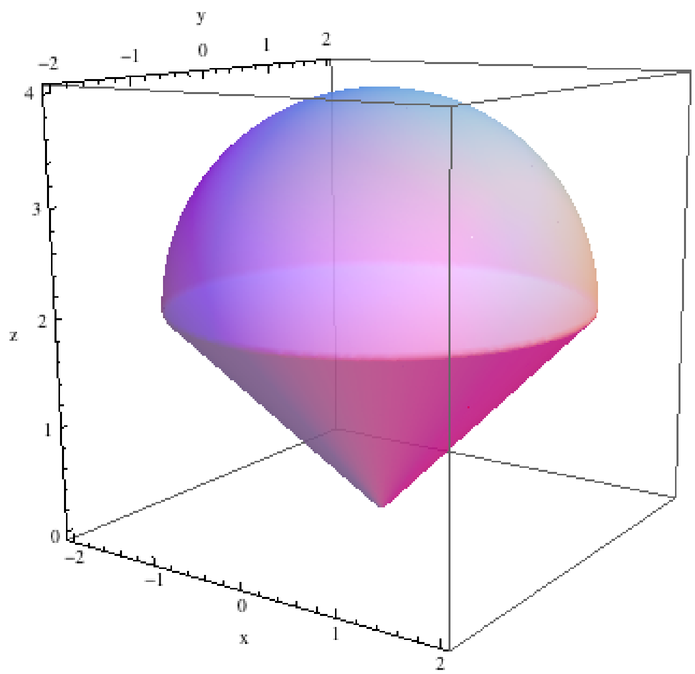
$$\begin{aligned} x^2 + y^2 \leq z &\leq 36 - 3x^2 - 3y^2 \\ r^2 \leq z &\leq 36 - 3r^2 \end{aligned}$$

Then we can solve the integral

$$\begin{aligned} \iiint_E dV &= \int_0^{2\pi} \int_0^3 \int_{r^2}^{36-3r^2} r\, dz\, dr\, d\theta = 2\pi \int_0^3 r(36 - 3r^2 - r^2)\, dr \\ &= 2\pi \int_0^3 36r - 4r^3\, dr = 2\pi \left[18r^2 - r^4\right]_0^3 \\ &= 2\pi[18(9) - 81] = \mathbf{162}\pi \end{aligned}$$

Example 4 (#29, §15.9): Find the volume of the solid that lies above the cone $\phi = \pi/3$ and below the sphere $\rho = 4\cos\phi$.

Solution. The shape will look like this:



Notice that $\rho = 4\cos\phi$, then $\rho^2 = 4\rho\cos\phi$, which is the equation of a sphere of radius 2 centered at (0,0,2), then we see $0 \leq \rho \leq 4\cos\phi$ while the cone indicates $0 \leq \phi \leq \pi/3$, and since the object is circular, $0 \leq \theta \leq 2\pi$, which allows us to calculate the integral:

$$\begin{aligned}
V &= \int_0^{2\pi} \int_0^{\pi/3} \int_0^{4\cos\phi} \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta \\
&= 2\pi \int_0^{\pi/3} \left[\frac{\rho^3}{3}\right]_{\rho=0}^{\rho=4\cos\phi} \sin\phi \, d\phi \\
&= \frac{2\pi}{3} \int_0^{\pi/3} 64\cos^3\phi \sin\phi \, d\phi \\
&= \frac{2\pi}{3} \left[-16\cos^4\phi\right]_0^{\pi/3} = -\frac{32\pi}{3}\left[\frac{1}{16} - 1\right] = \frac{32\pi}{3}\frac{15}{16} = 10\pi
\end{aligned}$$