1 Highlights (§15.7)

– Disclaimer: This is NOT a complete list of what you need to understand. Any material in the sections may appear on tests.

– Triple integrals are the extension of integrals to functions of 3 variables. Most ideas apply analogously in evaluating them, but now we integrate over 3-dimensional regions, which can be more difficult to set up.

– The ideas may be less intuitive in a physical sense. Triple integrals yield "hyper-volume," which is the generalization of volume to 4 dimensions, which is not something we can visualize effectively.

– Integrals in 3 (or more) dimensions are still very useful, and a few applications are shown in pages 1047-1048 of Stewart.

– We again have some classes of regions of integration. For some 3D region E, if D is the projection of E onto one of the 2D coordinate planes, then the types are

- Type I region: $E = \{(x, y, z) \mid (x, y) \in D, u_1(x, y) \leq z \leq u_2(x, y)\}$
  - $\iiint\limits_E f(x, y, z)\, dV = \iint\limits_D \left[\int_{u_1(x,y)}^{u_2(x,y)} f(x, y, z)\, dz\right] dA$
- Type II region: $E = \{(x, y, z) \mid (y, z) \in D, u_1(y, z) \leq x \leq u_2(y, z)\}$
  - $\iiint\limits_E f(x, y, z)\, dV = \iint\limits_D \left[\int_{u_1(y,z)}^{u_2(y,z)} f(x, y, z)\, dx\right] dA$
- Type III region: $E = \{(x, y, z) \mid (x, z) \in D, u_1(x, z) \leq y \leq u_2(x, z)\}$
  - $\iiint\limits_E f(x, y, z)\, dV = \iint\limits_D \left[\int_{u_1(x,z)}^{u_2(x,z)} f(x, y, z)\, dy\right] dA$

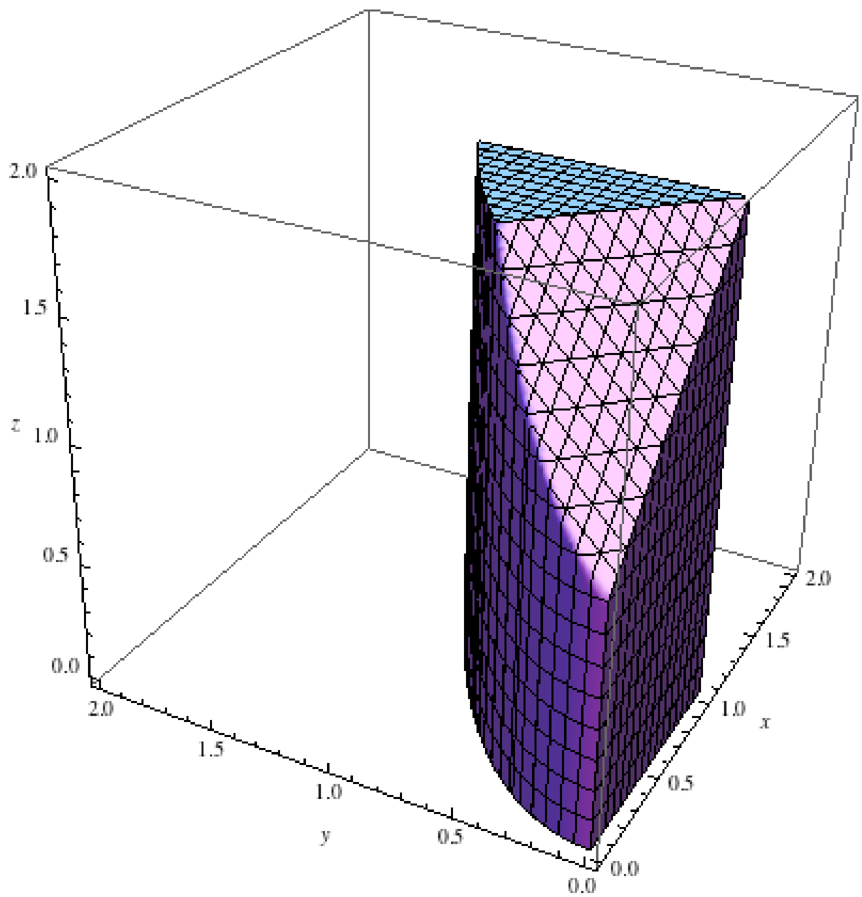
– Note that after we do the innermost integral, we still have a double integral, so we will need to figure out the bounds on D as in previous sections.

- Given the full information on D, we can write E more specifically and the integral more clearly.
- If $E = \{(x, y, z) \mid a \leq x \leq b, g_1(x) \leq y \leq g_2(x), u_1(x, y) \leq z \leq u_2(x, y)\}$, then
  - $\iiint\limits_E f(x, y, z)\, dV = \int_a^b \int_{g_1(x)}^{g_2(x)} \int_{u_1(x,y)}^{u_2(x,y)} f(x, y, z)\, dz\, dy\, dx$
- This is just one example, any ordering of the integrals is possible for different types of regions - #'s 29, 31, 33, 35 are good practice.

2 Problems

Example 1 (#13): Evaluate $\iiint_E 6xy\,dV$ where E lies under the plane $z = 1 + x + y$ and above the region in the xy-plane bounded by the curves $y = \sqrt{x}$, $y = 0$, and $x = 1$.

Solution. Since z is between the xy-plane and $1+x+y$, we have $0 \leq z \leq 1+x+y$. We also have $0 \leq y \leq \sqrt{x}$ and $0 \leq x \leq 1$. Our region then looks like this:



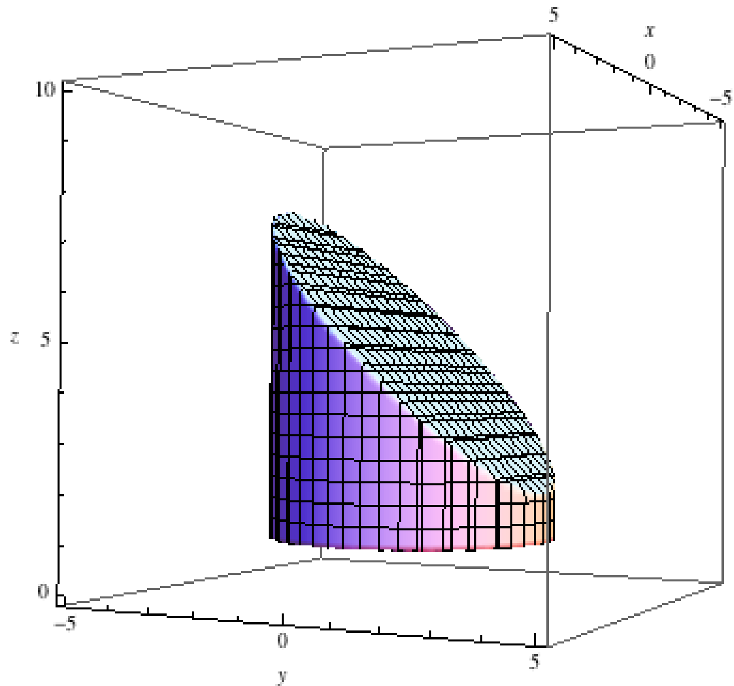
We can set up the integral as

$$
\begin{aligned}
\iiint_E 6xy\,dV &= \int_0^1 \int_0^{\sqrt{x}} \int_0^{1+x+y} 6xy\,dz\,dy\,dx \\
&= 6\int_0^1 x \int_0^{\sqrt{x}} y\,[z]_0^{1+x+y}\;dy\,dx \\
&= 6\int_0^1 x \int_0^{\sqrt{x}} y(1+x+y)\,dy\,dx \\
&= 6\int_0^1 x \int_0^{\sqrt{x}} y(1+x)+y^2\,dy\,dx \\
&= 6\int_0^1 x \left[(1+x)\frac{y^2}{2}+\frac{y^3}{3}\right]_0^{\sqrt{x}} dx \\
&= 6\int_0^1 x \left[(1+x)\frac{x}{2}+\frac{x^{3/2}}{3}\right] dx \\
&= 6\int_0^1 \frac{x^2}{2}+\frac{x^3}{2}+\frac{x^{5/2}}{3}\,dx \\
&= 6\left[\frac{x^3}{6}+\frac{x^4}{8}+\frac{2x^{7/2}}{21}\right]_0^1 \\
&= 6\left(\frac{1}{6}+\frac{1}{8}+\frac{2}{21}\right)=\frac{65}{28}
\end{aligned}
$$

Note that we didn't pull out the x like line 2 in class, but assuming we didn't make arithmetic or algebraic errors somewhere, the solution should be the same.

Example 2 Use a triple integral to find the volume of the solid enclosed by the cylinder $x^2+y^2=9$ and the planes $z=1$ and $y+z=5$.

Solution. Our solid is a vertical cylinder between a horizontal plane on the bottom and an angled plane at the top:



The lower plane is at 1, and the upper plane is $y + z = 5$, then we can see $1 \le z \le 5 - y$, and solving for y in the equation of the cylinder, $x^2 + y^2 = 9$, we find $-\sqrt{9 - x^2} \le y \le \sqrt{9 - x^2}$. Finally, since the radius of the cylinder is 3 and it's centered at $(0, 0)$ in the xy-plane, we see $-3 \le x \le 3$, so we can set up the integral:

$$\iiint\limits_E dV = \int_{-3}^{3} \int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} \int_{1}^{5-y} dz\, dy\, dx = \int_{-3}^{-3} \int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} 5 - y - 1\, dy\, dx$$

$$= \int_{-3}^{3} \int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} 4 - y\, dy\, dx = \int_{-3}^{3} \left[4y - \frac{y^2}{2}\right]_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} dx$$

$$= \int_{-3}^{3} 8\sqrt{9 - x^2} - \frac{9 - x^2 - 9 + x^2}{2}\, dx$$

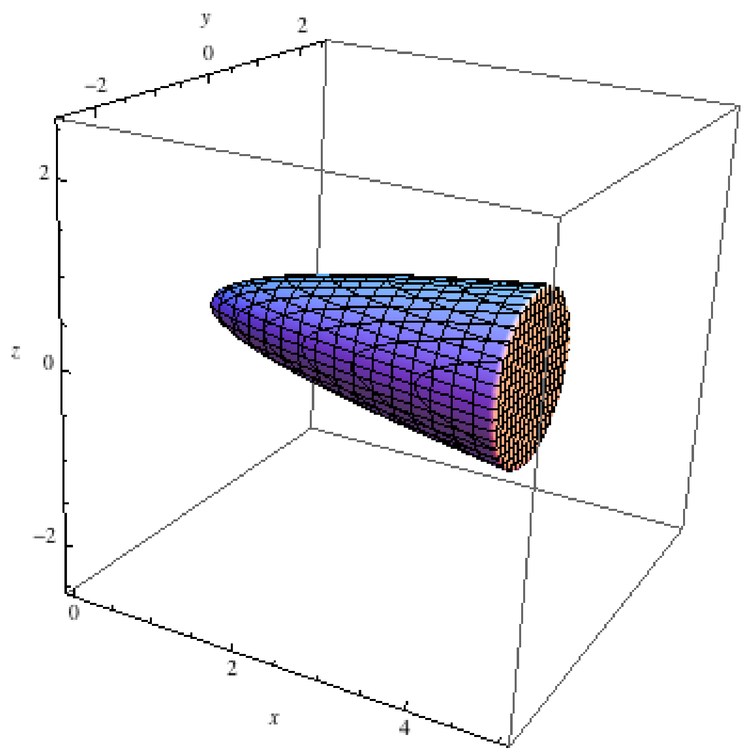
$$= 8 \int_{-3}^{3} \sqrt{9 - x^2}\, dx = 4 \left[x\sqrt{9 - x^2} + \frac{9}{2} \sin^{-1} \frac{x}{3}\right]_{-3}^{3}$$

$$= 36 \left[\sin^{-1}(1) - \sin^{-1}(-1)\right] = 18 \left[\frac{\pi}{2} + \frac{\pi}{2}\right] = 36\pi$$

Example 3 (#17): Evaluate $\iiint\limits_E x\, dV$, where E is bounded by paraboloid

$x = 4y^2 + 4z^2$ and the plane $x = 4$.

Solution. As we can see, x must be nonnegative given the equation of the paraboloid, and will be equal to 0 at the apex of the paraboloid, then we can see $4y^2 + 4z^2 \le x \le 4$.



Next, let D be the projection of E onto the yz-plane by considering its widest part (at $x = 4$), so the projection will be $4y^2 + 4z^2 \le 4$, or $y^2 + z^2 \le 1$ (i.e. a circle!). Using polar coordinates in the yz-plane, we can define $y = r\cos\theta$ and $z = r\sin\theta$, where we will bound r by 1 (the radius is 1) and every angle should be considered, $0 \le \theta \le 2\pi$. Then we have

$$\iiint_E x\,dV = \iint_D \int_{4y^2+4z^2}^{4} x\,dx\,dA = \iint_D \left[\frac{x^2}{2}\right]_{4y^2+4z^2}^{4} dA$$
$$= \iint_D \frac{1}{2}\left(16 - 16(y^2+z^2))^2\right) dA$$
$$= 8\int_0^{2\pi}\int_0^1 (1-r^4)r\,dr\,d\theta = 8\int_0^{2\pi} d\theta \int_0^1 r - r^5\,dr$$
$$= 16\pi\left[\frac{r^2}{2} - \frac{r^6}{6}\right]_0^1 = 16\pi\left[\frac{1}{2} - \frac{1}{6}\right] = \frac{16\pi}{3}$$

Example 4: Find the mass the cube given by $0 \le x \le a$, $0 \le y \le a$, and $0 \le z \le a$ with the density function $\rho(x,y,z) = x^2+y^2+z^2$

Solution. Since the region is a cube, the integral is simple to set up.

$$m = \int_0^a\int_0^a\int_0^a x^2+y^2+z^2\,dx\,dy\,dz = \int_0^a\int_0^a \left[\frac{x^3}{3} + (y^2+z^2)x\right]_0^a dy\,dz$$
$$= \int_0^a\int_0^a \frac{a^3}{3} + ay^2 + az^2\,dy\,dz = \int_0^a \left[\frac{a^3}{3}y + \frac{ay^3}{3} + az^2y\right]_0^a dz$$
$$= \int_0^a \frac{a^4}{3} + \frac{a^4}{3} + a^2z^2\,dz = \left[\frac{2a^4}{3}z + \frac{a^2z^3}{3}\right]_0^a = \frac{2a^5}{3} + \frac{a^5}{3} = a^5$$