§16.5 Curl and Divergence

- Disclaimer: This is NOT a complete list of what you need to understand, additional material in the text may appear on tests.
- Curl and divergence are operations performed on vector fields, both resembling differentiation in some sense.
- The curl operation yields a vector field indicating the tendency of particles to rotate about the axis pointing in the direction of curl $\mathbf{F}(x, y, z)$ at each point (x, y, z).
- Let $\nabla = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle$
- If $\mathbf{F} = P\mathbf{i}+Q\mathbf{j}+R\mathbf{k}$ is a vector field in $\mathbb{R}^3$ where the first partial derivatives of P, Q, and R exist, then

$$\begin{aligned}\text{curl } \mathbf{F} = \nabla \times \mathbf{F} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix} \\ &= \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z} \right) \mathbf{i} - \left(\frac{\partial R}{\partial x} - \frac{\partial P}{\partial z} \right) \mathbf{j} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \mathbf{k}\end{aligned}$$

  - If $\mathbf{F}$ is defined on all of $\mathbb{R}^3$ and its component functions have continuous partial derivatives, $\mathbf{F}$ is conservative if and only if curl $\mathbf{F} = \mathbf{0}$.

- The divergence operation yields a scalar field measuring the tendency of particles to move away from each point (x, y, z).
- If $\mathbf{F} = P\mathbf{i}+Q\mathbf{j}+R\mathbf{k}$ is a vector field in $\mathbb{R}^3$ where the first partial derivatives of P, Q, and R exist, then

$$\text{div } \mathbf{F} = \nabla \cdot \mathbf{F} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}$$

- Curl and divergence lead to another form of Green's Theorem.
  - Let $C = \mathbf{r}(t)$ be a curve enclosing D satisfying the conditions of Green's Theorem.
  - Recall $\mathbf{n}(t) = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|}$ is the unit normal vector of a curve C, then

$$\oint_C \mathbf{F} \cdot \mathbf{n} \, ds = \iint_D \text{div } \mathbf{F}(x, y) \, dA$$

  The left side is read as "the line integral of the normal component of $\mathbf{F}$ along C."

§16.6 Parametric Surfaces and Their Areas

– For $(u, v) \in D$ for some domain D, a parametric surface is defined by a vector function of two variables,

$$\mathbf{r}(u, v) = x(u, v)\mathbf{i} + y(u, v)\mathbf{j} + z(u, v)\mathbf{k} \tag{1}$$

– Parametric surfaces allow us to define a very general class of surfaces, as opposed to only the very specific surfaces we have seen in the past like spheres and cylinders.

– If a smooth parametric surface S is given by equation (1) for $(u, v) \in D$ and S is covered just once as (u, v) ranges through D, then the surface area of S is

$$A(S) = \iint_D |\mathbf{r}_u \times \mathbf{r}_v| \, dA$$

Problems

Example 1 (§16.5 #15): Is $\mathbf{F}(x, y, z) = 2xy\mathbf{i} + (x^2 + 2yz)\mathbf{j} + y^2\mathbf{k}$ conservative? If so, find a function f such that $\mathbf{F} = \nabla f$.

Solution. Since the P, Q, and R terms are all polynomials, they have continuous partial derivatives, so if curl $\mathbf{F} = \mathbf{0}$, then $\mathbf{F}$ is conservative.

$$\text{curl } \mathbf{F} = (2y - 2y)\mathbf{i} - (0 - 0)\mathbf{j} + (2x - 2x)\mathbf{k} = \mathbf{0}$$

Therefore, $\mathbf{F}$ is conservative. Next, we need to find f such that $\mathbf{F} = \nabla f = f_x\mathbf{i} + f_y\mathbf{j} + f_z\mathbf{k}$. Suppose $f_x = P = 2xy$, then if we integrate with respect to x, we find

$$f(x, y, z) = x^2y + g(y, z) \tag{2}$$

The constant of integration here is a function of y and z. Differentiating with respect to y, we find $f_y = x^2 + g_y(y, z)$, but we should have $Q = f_y$, then we must have $g_y(y, z) = 2yz$.

Finally, we need to make sure $R = f_z$. Integrating f_y with respect to y, we find

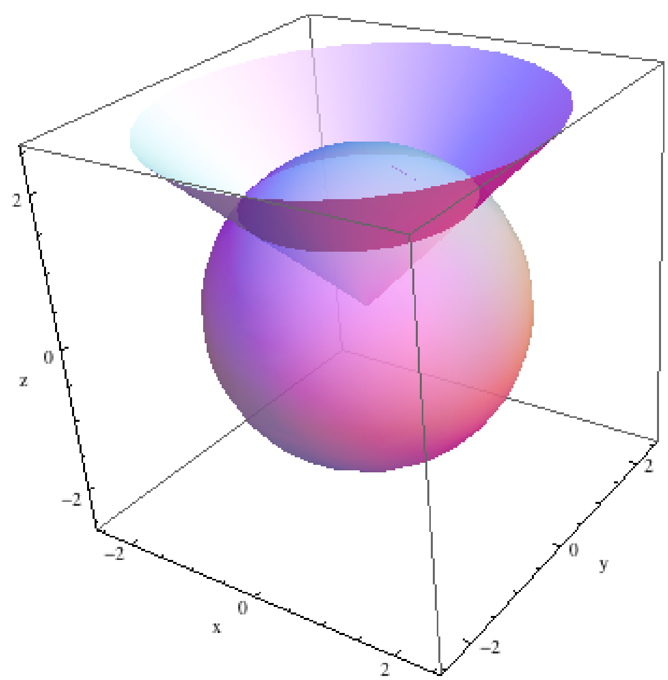
$$f(x, y, z) = \int x^2 + 2yz \, dy = x^2y + y^2z + h(z) \tag{3}$$

The constant of integration depends on z, then differentiating this latest f with respect to z, we find $f_z = y^2 + h'(z)$, then $h'(z) = 0$, so $h(z) = C$ for some constant C.

Thus, we have $f(x, y, z) = x^2y + y^2z + C$.

Example 2 (§16.6 #23): Find the parametric representation of the part of the sphere $x^2 + y^2 + z^2 = 4$ that lies above the cone $z = \sqrt{x^2 + y^2}$. Then find its surface area.

Solution. We are considering the surface the part of the sphere above the cone:



The cone and sphere intersect along a circle, and we will want to bound x and y within the projection of this circle on the xy-plane. Let's set the equations equal to one another to find projection of the intersection:

$$\begin{aligned} z^2 &= z^2 \\ x^2 + y^2 &= 4 - x^2 - y^2 \quad \text{(use the cone on the left and sphere on the right)} \\ 2(x^2 + y^2) &= 4 \\ x^2 + y^2 &= 2 \quad \text{(a circle of radius } \sqrt{2} \text{ centered at the origin)} \end{aligned}$$

The z-coordinate of the intersection will simply be $z = \sqrt{x^2 + y^2} = \sqrt{2}$.

If we restrict x and y to within the circle and use the parametrization $x = u$ and $y = v$, we next get the z-coordinate of the sphere using its equation: $z = \sqrt{4 - u^2 - v^2}$ for $u^2 + v^2 \leq 2$.

To find the surface area, we can plug in our parametrization to find $\mathbf{r}(u,v) = u\mathbf{i} + v\mathbf{j} + \sqrt{4-u^2-v^2}\mathbf{k}$, then we have $\mathbf{r}_u = \mathbf{i} - \frac{u}{\sqrt{4-u^2-v^2}}\mathbf{k}$ and $\mathbf{r}_v = \mathbf{j} - \frac{v}{\sqrt{4-u^2-v^2}}\mathbf{k}$. Then we can use the formula for surface area:

$$A(S) = \iint_D |\mathbf{r}_u \times \mathbf{r}_v| \, dA$$

First, we need the cross product:

$$\begin{aligned}\mathbf{r}_u \times \mathbf{r}_v &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & -\frac{u}{\sqrt{4-u^2-v^2}} \\ 0 & 1 & -\frac{v}{\sqrt{4-u^2-v^2}} \end{vmatrix} \\ &= \frac{u}{\sqrt{4-u^2-v^2}}\mathbf{i} + \frac{v}{\sqrt{4-u^2-v^2}}\mathbf{j} + \mathbf{k}\end{aligned}$$

Then we need the magnitude of this vector, which is

$$\begin{aligned}|\mathbf{r}_u \times \mathbf{r}_v| &= \sqrt{\frac{u^2}{4-u^2-v^2} + \frac{v^2}{4-u^2-v^2} + 1} \\ &= \sqrt{\frac{u^2+v^2+4-u^2-v^2}{4-u^2-v^2}} \\ &= \sqrt{\frac{4}{4-u^2-v^2}} = 2\sqrt{\frac{1}{4-u^2-v^2}}\end{aligned}$$

Plugging this back into the integral, we have

$$\begin{aligned}A(S) &= 2\iint_D \sqrt{\frac{1}{4-u^2-v^2}} \, dA \\ &= 2\int_0^{2\pi}\int_0^{\sqrt{2}} \frac{r}{\sqrt{4-r^2}} \, dr \, d\theta && \text{(using polar coordinates)} \\ &= 4\pi \int_0^{\sqrt{2}} \frac{r}{\sqrt{4-r^2}} \, dr && \text{(the integrand is independent of } \theta\text{)} \\ &= -2\pi \int_4^2 \frac{1}{\sqrt{w}} \, dw && \text{(substitution } w = 4-r^2,\ dw = -2r\,dr\text{)} \\ &= 2\pi \int_2^4 w^{-1/2} \, dw && \text{(switch the order of the bounds)} \\ &= 2\pi \left[2w^{1/2}\right]_2^4 \\ &= \mathbf{4}\left(\mathbf{2} - \sqrt{\mathbf{2}}\right)\pi \approx \mathbf{2.434}\pi\end{aligned}$$

$$\nu_{p(1)} = ... = \nu_{p(s_1)} < \nu_{p(s_1+1)} = ... = \nu_{p(s_2)} < ... < \nu_{p(s_{k-1}+1)} = ... = \nu_{p(d)}$$

j_1

j_2

j_k

$\Big[\quad I_0 \quad \Big)$

$\Big[\quad I_1 \quad \Big)$

$\Big[\quad I_2$

t_1

t_2

t_3

t_4

t_5

t_6

t_7