§16.7 Surface Integrals

- Surface integrals are calculated as

$$\iint_S f(x,y,z)\,dS = \iint_D f(\mathbf{r}(u,v))|\mathbf{r}_u \times \mathbf{r}_v|\,dA$$

  where $\mathbf{r}(u,v)$ is the parameterization of the surface S.

- If we can write $z = g(x,y)$, this reduces to

$$\iint_S f(x,y,z)\,dS = \iint_D f(x,y,g(x,y))\sqrt{\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + 1}$$

- If $\mathbf{F}$ is a continuous vector field defined on an oriented surface S with unit normal vector $\mathbf{n}$ and $\mathbf{r}(u,v)$ is a parameterization of S, then the surface integral of $\mathbf{F}$ over S is

$$\iint_S \mathbf{F}\cdot\, d\mathbf{S} = \iint_S \mathbf{F}\cdot\mathbf{n}\,dS = \iint_D \mathbf{F}\cdot(\mathbf{r}_u \times \mathbf{r}_v)\,dA$$

  This is also known as the flux of $\mathbf{F}$ across S.

- If we can write $z = g(x,y)$, this reduces to

$$\iint_S \mathbf{F}\cdot d\mathbf{S} = \iint_D (-Pg_x - Qg_y + R)\;dA$$

  where $\mathbf{F} = \langle P, Q, R\rangle$

§16.8 Stokes' Theorem

- Let S be a piecewise-smooth oriented surface bounded by a simple, closed, piecewise-smooth curve C with positive orientation. If $\mathbf{F}$ is a vector field whose components have continuous partial derivatives on an open region in $\mathbb{R}^3$ containing S, then

$$\int_C \mathbf{F}\cdot\, d\mathbf{r} = \iint_S \text{curl }\mathbf{F}\cdot\, d\mathbf{S}$$

- Note that we can apply the simplification from the last item in the notes for §16.7 where curl $\mathbf{F} = \langle P, Q, R\rangle$

- If S_1 and S_2 are oriented surfaces bounded by the same curve C and the conditions of Stokes' Theorem are satisfied, then

$$\iint_{S_1} \text{curl }\mathbf{F}\cdot\, d\mathbf{S} = \int_C \mathbf{F}\cdot d\mathbf{r} = \iint_{S_2} \text{curl }\mathbf{F}\cdot d\mathbf{S}$$

§16.9 The Divergence Theorem

- Let E be a simple solid region bounded by a surface S with positive orientation. If $\mathbf{F}$ is a vector field whose components have continuous partial derivatives on an open region that contains S, then

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iiint_E \text{div } \mathbf{F} \, dV$$

Problems

Find the parametric representation of the surface S, which is the part of the sphere $x^2 + y^2 + z^2 = 1$ that lies above the cone $z = \sqrt{x^2 + y^2}$. Then, find the surface integral of $f(x, y, z) = z$ along S.