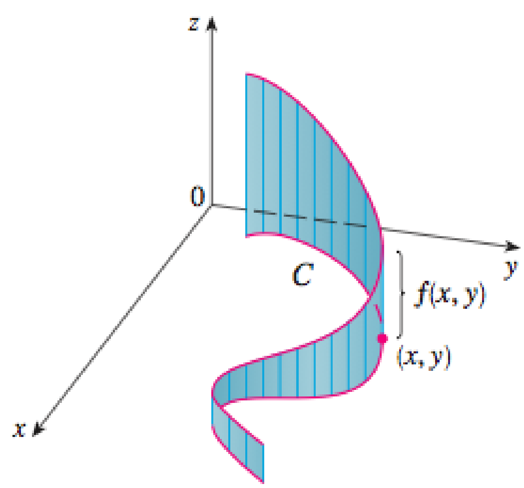
§16.1 Vector Fields

- Disclaimer: This is NOT a complete list of what you need to understand, additional material in the text may appear on tests.
- A vector field in a space region D in $\mathbb{R}^3$ maps each point (x, y, z) in D to a vector $\mathbf{F}(x, y, z)$.
- If f is a scalar function then ∇f is a **gradient vector field** (if it exists).
- A vector field $\mathbf{F}$ is **conservative** if it is the gradient of some scalar function f (the **potential function** of $\mathbf{F}$).
- The main skill of this section is determining if a vector field $\mathbf{F}$ is conservative and finding its potential function.

§16.2 Line Integrals

- Line integrals (or path integrals) are much like ordinary integrals, except they are taken over a curve C instead of an interval on the x-axis:



The line integral $\int_C f(x, y)\, ds$ represents the area under f along C.

- Let C be a smooth curve defined by parametric equations $\langle x(t), y(t), z(t)\rangle$ for $a \leq t \leq b$. The **line integral of a function f along C** is

$$\int_C f(x, y, z)\, ds = \int_a^b f\left(x(t), y(t), z(t)\right) \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\, dt$$

- This animation communicates the idea well.

– Suppose C is only piece-wise smooth: i.e. C is a union of some paths $C_1, ..., C_n$, each of which is smooth, then

$$\int_C f\, ds = \int_{C_1} f\, ds + \cdots + \int_{C_n} f\, ds$$

– The line integral of $f(x, y, z)$ along C with respect to x is

$$\int_C f(x, y, z)\, dx = \int_a^b f(x(t), y(t), z(t))x'(t)\, dt$$

– (**Line Integrals of Vector Fields**) If $\mathbf{F}$ is a continuous vector field defined on a smooth curve C defined by $\mathbf{r}(t)$, $a \leq t \leq b$. The line integral of $\mathbf{F}$ along C is

$$\int_C \mathbf{F} \cdot\, d\mathbf{r} = \int_a^b \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\, dt$$

- If $\mathbf{F}$ is a force field, $\int_C \mathbf{F} \cdot d\mathbf{r}$ represents the work done by the force to move a particle along path C.
- A great explanation/animation can be found here.

– We often must parametrize curves to find $\mathbf{r}(t)$.

§16.3 Fundamental Theorem of Line Integrals

– (**The Fundamental Theorem of Line Integrals**) Suppose C is a smooth curve given by $\mathbf{r}(t)$, $a \leq t \leq b$ and let $\mathbf{F}$ be a gradient field (i.e. $\mathbf{F} = \nabla f$ for some function f) where ∇f is continuous on C. Then

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C \nabla f \cdot d\mathbf{r} = f(\mathbf{r}(b)) - f(\mathbf{r}(a))$$

- This offers a shortcut for calculating line integrals if $\mathbf{F}$ is a gradient field (i.e. a conservative field).
- The Fundamental Theorem of Calculus is $\int_a^b F'(x)\, dx = F(b) - F(a)$ where $F'(x)$ is continuous on $[a, b]$. Notice this one is analogous.
- Notice the integral of a gradient field is *independent of the path C*: it depends only on the start and endpoints, $\mathbf{r}(a)$ and $\mathbf{r}(b)$.

– **Theorem**: If $\mathbf{F}(x, y) = P(x, y)\mathbf{i} + Q(x, y)\mathbf{j}$ is conservative where P and Q have continuous partial derivatives on D, then in D,

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$$

- The converse is only true under some extra assumptions.

– **Theorem**: Let $\mathbf{F}$ be defined on an open, simply-connected region D. If P and Q have continuous partial derivatives and

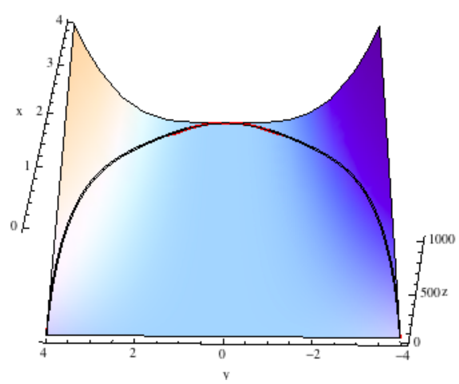
$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$$

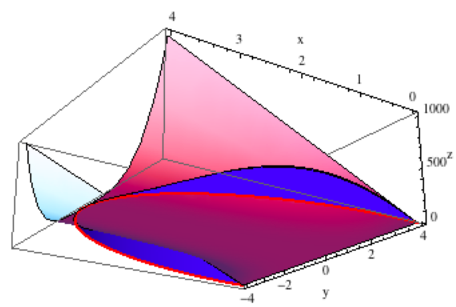
then $\mathbf{F}$ is conservative.

Problems

Example 1 (16.2, #3): Evaluate $\int_C xy^4 \, ds$ where C is the right half of the circle $x^2 + y^2 = 16$.

Solution. Before solving, let's think about what the integrals represents. We have some surface $z = xy^4$, and a circle on the xy-plane, $x^2 + y^2 = 16$. The integral $\int_C xy^4 \, ds$ will be the area under the surface along that curve.





The first picture shows curve C marked along the surface and the second is a view from below, showing the filling between the curve on the xy-plane and the surface.

Now, to solve it, notice $x^2+y^2 = 16$ is a circle centered at $(0, 0)$ with radius 4 and recall that one parametric representation of a circle of radius 1 is $x(t) = \cos(t)$, $y(t) = \sin(t)$ with $0 \le t \le 2\pi$.

To make a larger circle, we just multiply each by 4 to get $x(t) = 4\cos(t)$, $y(t) = 4\sin(t)$, and choose the angles representing the right half of the circle, so we can use $-\frac{\pi}{2} \le t \le \frac{\pi}{2}$ (note that the bounds on t may vary for different parameterizations of the right half of the circle).

Plugging in these details, we can integrate:

$$\begin{aligned}\int_C xy^4\,ds &= 4^5\int_{-\pi/2}^{\pi/2}\cos(t)\sin^4(t)\sqrt{(-4\sin(t))^2+(4\cos(t))^2}\,dt\\ &= 4^6\int_{-\pi/2}^{\pi/2}\cos(t)\sin^4(t)\,dt = 4^6\int_{-1}^{1}u^4\,du\\ &= 4^6\left[\frac{u^5}{5}\right]_{-1}^{1} = 4^6\left(\frac{2}{5}\right) = \frac{2^{13}}{5}\approx \mathbf{1638.4}\end{aligned}$$

Example 2 (16.2, # 11): Evaluate $\int_C xe^{yz}\,ds$ where C is the line segment from $(0,0,0)$ to $(1,2,3)$.

Solution. We cannot quite draw this one since it's a function with 3 inputs, but we can solve it analogously nevertheless.

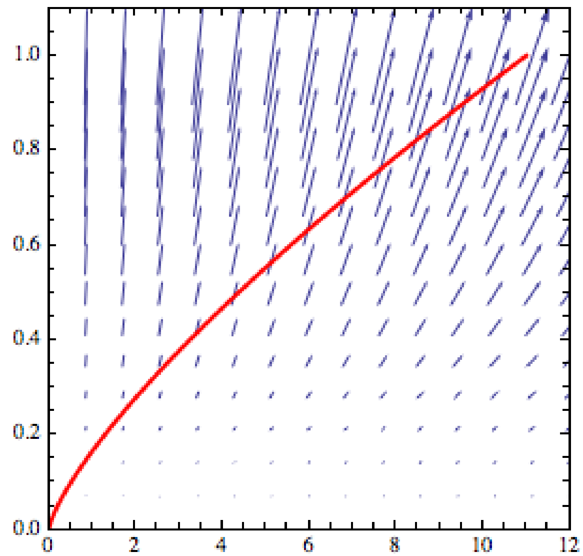
First, we find a parametric representation of the vector function $\mathbf{r}(t)$ representing the required line segment, one of which is $\langle t, 2t, 3t\rangle$ for $0 \le t \le 1$. So we can now solve the integral

$$\begin{aligned}\int_C xe^{yz}\,ds &= \int_0^1 te^{2t\cdot 3t}\sqrt{1^2+2^2+3^2}\,dt = \sqrt{14}\int_0^1 te^{6t^2}\,dt\\ &= \sqrt{14}\int_0^6 \frac{1}{12}e^u\,du = \sqrt{14}\left[\frac{e^u}{12}\right]_0^6 = \frac{\sqrt{\mathbf{14}}}{\mathbf{12}}(\mathbf{e^6}-\mathbf{1})\end{aligned}$$

Note that we made the substitution $u = 6t^2$, $du = 12t\,dt$.

Example 3 (16.2, # 19) Evaluate the integral of $\mathbf{F}(x,y) = xy\mathbf{i} + 3y^2\mathbf{j}$ along the curve $\mathbf{r}(t) = 11t^4\mathbf{i} + t^3\mathbf{j}$, $0 \le t \le 1$.

Solution. We will find a line integral over the vector field $\mathbf{F}$ on this problem, which if $\mathbf{F}$ is a force field, would represent the amount of work required for a particle to move along the curve $\mathbf{r}(t)$ from $t = 0$ to $t = 1$ (which is from the lower left to the upper right on this curve):

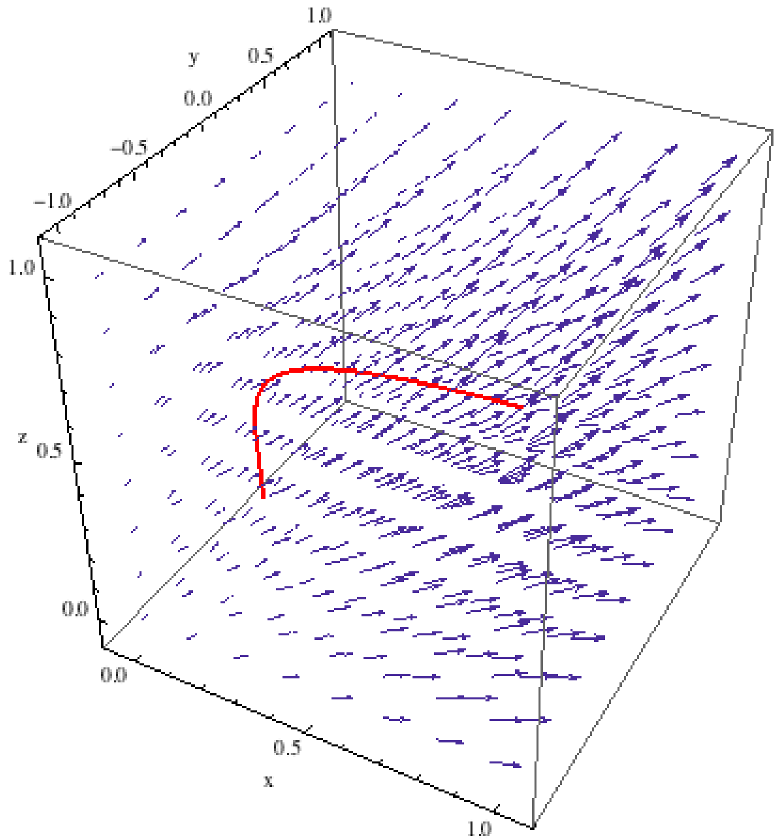


First, we find $\mathbf{r}'(t) = 44t^3\mathbf{i} + 3t^2\mathbf{j}$, and so we can integrate via the formula above.

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^1 \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\, dt = \int_0^1 \langle 11t^7, 3t^6 \rangle \cdot \langle 44t^3, 3t^2 \rangle\, dt$$
$$= \int_0^1 (484t^{10} + 9t^8)\, dt = \left[44t^{11} + t^9\right]_0^1 = \mathbf{45}$$

Example 4 (16.2, #21): Evaluate the integral of $\mathbf{F}(x,y,z) = \sin(x)\mathbf{i} + \cos(y)\mathbf{j} + xz\mathbf{k}$ along the curve $\mathbf{r}(t) = t^3\mathbf{i} - t^2\mathbf{j} + t\mathbf{k}$, $0 \le t \le 1$.

Solution. We will find a line integral over $\mathbf{F}$ over $\mathbf{r}(t)$:

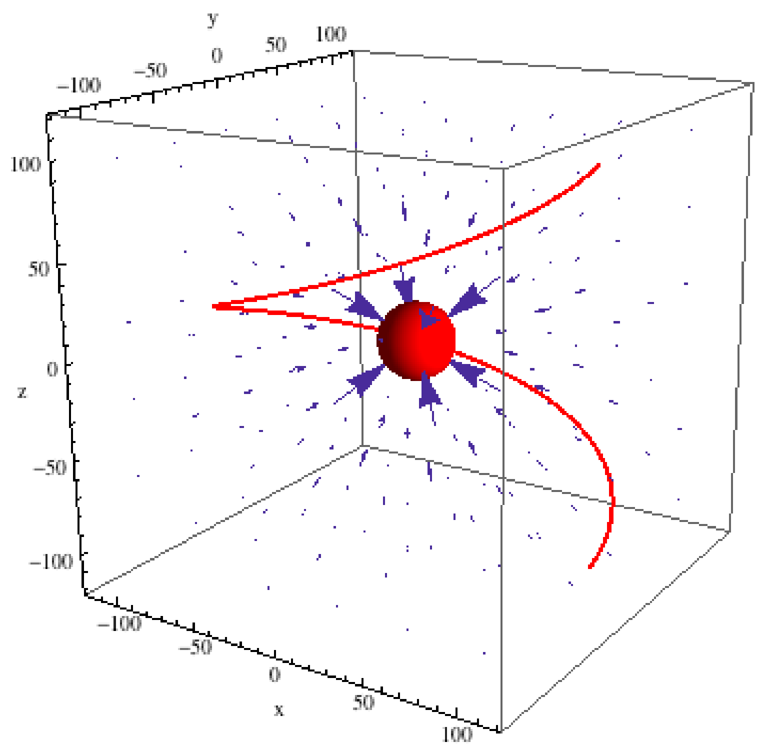


First, we need the derivative of the path, $\mathbf{r}'(t) = 3t^2\mathbf{i} - 2t\mathbf{j} + \mathbf{k}$, and we can integrate by plugging in $\mathbf{F}(\mathbf{r}(t))$:

$$\begin{aligned}
\int_C \mathbf{F} \cdot \, d\mathbf{r} &= \int_0^1 \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\, dt \\
&= \int_0^1 \langle \cos(t^3), \sin(-t^2), t^4 \rangle \cdot \langle 3t^2, -2t, 1 \rangle \, dt \\
&= \int_0^1 3t^2 \cos(-t^3) - 2t \sin(-t^2) + t^4 \, dt \\
&= \left[-\sin(t^3) - \cos(-t^2) + \frac{1}{5}t^5 \right]_0^1 \\
&= -\sin 1 - \cos -1 + \frac{1}{5} + 1 \\
&= \mathbf{\frac{6}{5} - \sin\ 1 - \cos\ (-1)}
\end{aligned}$$

Example 5 Suppose we have a planet of mass M and propel a spacecraft of mass m one revolution and up in the z direction along a left-handed helical path where the radius of the projection of the path onto the xy-plane is R, and up h units in the z direction. How much work would be done? [Hint: place the center of the planet at $(0, 0, 0)$]

Solution 1 (Direct Line Integral Calculation). We seek the amount of work to propel a spacecraft along some path amid the gravitational pull from the planet, so our goal must be a line integral of the vector field representing the gravitational force along the desired path (image with $R = 100, h = 100$):



In §16.1, Example 4, we are given the gravitational field (assuming the planet is the only substantive source of gravitation) via the inverse square law as

$$\mathbf{F}(x, y, z) = \frac{-mMGx}{(x^2 + y^2 + z^2)^{3/2}}\mathbf{i} + \frac{-mMGy}{(x^2 + y^2 + z^2)^{3/2}}\mathbf{j} + \frac{-mMGz}{(x^2 + y^2 + z^2)^{3/2}}\mathbf{k}$$

Next, we need to find a vector function $\mathbf{r}(t)$ representing the path along which the craft will travel. The helix will have x and y coordinates traversing a circle of radius R while z will rise steadily from $-\frac{h}{2}$ and $\frac{h}{2}$.

The helix we want is left-handed, but since gravity will be symmetric about the planet, it doesn't matter which way our parameterization of the helix goes, so we may arbitrarily set $x = R\cos(t)$, $y = R\sin(t)$, and one revolution will occur for $0 \leq t \leq 2\pi$.

Since $z(t)$ should rise steadily, it will be a linear function, where $z(0) = -h/2$ and $z(2\pi) = h/2$, so we find

$$\begin{aligned} z(t) - z(0) &= \frac{z(2\pi) - z(0)}{2\pi - 0}(t - 0) \\ z(t) + \frac{h}{2} &= \frac{h}{2\pi}t \\ z(t) &= \left(\frac{h}{2\pi}\right)t - \frac{h}{2} \end{aligned}$$

So we now have our path

$$C: \mathbf{r}(t) = R\cos(t)\mathbf{i} + R\sin(t)\mathbf{j} + \left[\left(\frac{h}{2\pi}\right)t - \frac{h}{2}\right]\mathbf{k}$$

Given some values for R, h, m, M, G, we can plot the path around the planet and the gravitational field

We also need the derivative of the path,

$$\mathbf{r}'(t) = -R\sin(t)\mathbf{i} + R\cos(t)\mathbf{j} + \left(\frac{h}{2\pi}\right)\mathbf{k}$$

We will use $\mathbf{F}(\mathbf{r}(t))$, which we can find

$$\begin{aligned} \mathbf{F}(\mathbf{r}(t)) = &\frac{-mMGR\cos(t)}{\left(R^2 + \left[\left(\frac{h}{2\pi}\right)t - \frac{h}{2}\right]^2\right)^{3/2}}\mathbf{i} + \frac{-mMGR\sin(t)}{\left(R^2 + \left[\left(\frac{h}{2\pi}\right)t - \frac{h}{2}\right]^2\right)^{3/2}}\mathbf{j} \\ &+ \frac{-mMGR\left[\left(\frac{h}{2\pi}\right)t - \frac{h}{2}\right]}{\left(R^2 + \left[\left(\frac{h}{2\pi}\right)t - \frac{h}{2}\right]^2\right)^{3/2}}\mathbf{k} \end{aligned}$$

Within the integral, we will do $\mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)$:

$$\begin{aligned} \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t) &= \frac{mMGR^2\cos(t)\sin(t)}{\left(R^2 + \left[\left(\frac{h}{2\pi}\right)t - \frac{h}{2}\right]^2\right)^{3/2}} + \frac{-mMGR^2\cos(t)\sin(t)}{\left(R^2 + \left[\left(\frac{h}{2\pi}\right)t - \frac{h}{2}\right]^2\right)^{3/2}} \\ &+ \frac{-\left(\frac{h}{2\pi}\right)mMGR\left[\left(\frac{h}{2\pi}\right)t - \frac{h}{2}\right]}{\left(R^2 + \left[\left(\frac{h}{2\pi}\right)t - \frac{h}{2}\right]^2\right)^{3/2}} \\ &= \left(\frac{-mMGRh}{2\pi}\right)\frac{\left[\left(\frac{h}{2\pi}\right)t - \frac{h}{2}\right]}{\left(R^2 + \left[\left(\frac{h}{2\pi}\right)t - \frac{h}{2}\right]^2\right)^{3/2}} \end{aligned}$$

Finally, we integrate this term to find the work required!

$$\begin{aligned}
W = \int_C \mathbf{F} \cdot\, d\mathbf{r} = \int_0^{2\pi} \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\, dt && (1)\\
= \frac{-mMGRh}{2\pi} \int_0^{2\pi} \frac{\left[\left(\frac{h}{2\pi}\right)t - \frac{h}{2}\right]}{\left(R^2 + \left[\left(\frac{h}{2\pi}\right)t - \frac{h}{2}\right]^2\right)^{3/2}}\, dt && (2)\\
= \frac{-mMGRh}{2\pi}\frac{\pi}{h} \int u^{-3/2}\, du = \frac{-mMGR}{2}\left[-2u^{-1/2}\right]_a^b && (3)\\
= mMGR\left[\frac{1}{\sqrt{R^2 + \left[\left(\frac{h}{2\pi}\right)t - \frac{h}{2}\right]^2}}\right]_0^{2\pi}\\
= mMGR\left[\frac{1}{\sqrt{R^2 + \left[h - \frac{h}{2}\right]^2}} - \frac{1}{\sqrt{R^2 + \left[-\frac{h}{2}\right]^2}}\right]\\
= 0
\end{aligned}$$

In line (3), we used the substitution $u = R^2 + \left[\left(\frac{h}{2\pi}\right)t - \frac{h}{2}\right]^2$, $du = \frac{h}{\pi}\left[\left(\frac{h}{2\pi}\right)t - \frac{h}{2}\right]\, dt$.

While this is a worthwhile exercise to practice setting up evaluating line integrals of vector fields, we will find in §16.3 that there is a much easier way to solve this problem since the gravitational field is a **conservative** vector field.

However, many fields are not conservative and must be approached this way.

Solution 2 (Fundamental Theorem of Line Integrals) Also given in §16.1 is that $\mathbf{F}$ is the gradient of $f(x, y, z) = -\frac{mMG}{\sqrt{x^2+y^2+z^2}}$. Since $\nabla f = \mathbf{F}$ is continuous and the path taken by the spacecraft is smooth, the Fundamental Theorem of Line Integrals implies

$$\begin{aligned}
W &= \int_C \mathbf{F} \cdot d\mathbf{r}\\
&= f(\mathbf{r}(2\pi)) - f(\mathbf{r}(0))\\
&= f\left(R\cos 2\pi, R\sin 2\pi, \left(\frac{h}{2\pi}\right)2\pi - \frac{h}{2}\right) - f\left(R\cos 0, R\sin 0, \left(\frac{h}{2\pi}\right)(0) - \frac{h}{2}\right)\\
&= f\left(R, 0, \frac{h}{2}\right) - f\left(R, 0, -\frac{h}{2}\right)\\
&= -\frac{mMG}{\sqrt{R^2 + \frac{h^2}{4}}} + \frac{mMG}{\sqrt{R^2 + \frac{h^2}{4}}}\\
&= 0
\end{aligned}$$