Chapter 16 Notes

Disclaimer: This is NOT a complete list of what you need to understand, additional ideas could appear on tests, but these are the most important.

Knowing the formulas and theorems is only part of the challenge. Doing problems efficiently requires an ability to determine appropriate solution methods – practicing problems is the best way to prepare.

Do odd problems so you can check your answers in the back of the book. If you start on a problem and find you're not sure what to do, try to find a similar problem in the examples in the book or in lab notes and see where you're going wrong.

Come to my your professor's office hours or e-mail me if you get stuck, we are happy to assist!

§16.1 Vector Fields

– A vector field $\mathbf{F}$ is called **conservative** if it is the gradient of some scalar function f, i.e. if there exists some f such that $\mathbf{F} = \nabla f$.

§16.2 Line Integrals

Good practice problems: 1-16, 19-30 (+ review parameterizing space curves)

– Let C be a plane curve defined by parametric equations $\mathbf{r}(t)$ for $a \leq t \leq b$, then the line integral of a function f along C is

$$\int_C f(x, y, z)\, ds = \int_a^b f\left(\mathbf{r}(t)\right) \|\mathbf{r}(t)\|\, dt$$

- This animated GIF communicates the idea well.

– If $\mathbf{F}$ is a continuous vector field defined on a smooth curve C defined by $\mathbf{r}(t)$, $a \leq t \leq b$. The line integral of $\mathbf{F}$ along C is

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_a^b \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\, dt$$

- A great explanation/animation can be found here.

§16.3 Fundamental Theorem of Line Integrals

Good practice problems: 3-10, 12-20, 23-24, 31-34

– Let C be a smooth curve given by the vector function $\mathbf{r}(t)$, $a \leq t \leq b$. Let f be a differentiable function of two or three variables whose gradient vector ∇f is continuous on C. Then

$$\int_C \nabla f \cdot d\mathbf{r} = f(\mathbf{r}(b)) - f(\mathbf{r}(a))$$

– This allows us to evaluate the line integral of a conservative vector field simply by knowing the value of f at the endpoints of C, which is much easier than previous methods.

– Let $\mathbf{F} = P\mathbf{i} + Q\mathbf{j}$ be a vector field on an open simply connected region D. Suppose that P and Q have continuous first-order derivatives and $P_y = Q_x$ throughout D, then $\mathbf{F}$ is conservative.

§16.4 Green's Theorem

Good practice problems: 1-16 (+ review double integrals)

– Green's Theorem gives the relationship between line integrals around a simple closed curve C and a double integral over the plane region D which lies inside C.

– **Green's Theorem**: Let C be a positively oriented, piecewise-smooth, simple closed curve in the plane and let D be the region bounded by C. If P and Q have continuous partial derivatives on the open region that contains D, then

$$\int_C P\,dx + Q\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA \tag{1}$$

- $C = \partial D$ and $\int_C P\,dx + Q\,dy = \oint P\,dx + Q\,dy$ are alternate notations

– The above version of Green's Theorem requires D to be simple, but it can work on regions that are finite unions of simple regions and regions that are not simply-connected (i.e. containing holes) – see pages 1111-1113

§16.5 Curl and Divergence

Good practice problems: 1-8, 13-18

– Curl and divergence are operations performed on vector fields, both resembling differentiation in some sense.

– The curl operation yields a **vector** field indicating the tendency of particles to rotate about the axis pointing in the direction of curl $\mathbf{F}(x, y, z)$ at each point (x, y, z).

– Let $\nabla = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle$

– If $\mathbf{F} = P\mathbf{i} + Q\mathbf{j} + R\mathbf{k}$ is a vector field in $\mathbb{R}^3$ where the first partial derivatives of P, Q, and R exist, then

$$\begin{aligned} \text{curl}\,\mathbf{F} = \nabla \times \mathbf{F} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix} \\ &= \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z} \right) \mathbf{i} - \left(\frac{\partial R}{\partial x} - \frac{\partial P}{\partial z} \right) \mathbf{j} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \mathbf{k} \end{aligned}$$

- If $\mathbf{F}$ is defined on all of $\mathbb{R}^3$ and its component functions have continuous partial derivatives, $\mathbf{F}$ is conservative if and only if curl $\mathbf{F} = \mathbf{0}$.

– The divergence operation yields a **scalar** field measuring the tendency of particles to move away from each point (x, y, z).

– If $\mathbf{F} = P\mathbf{i} + Q\mathbf{j} + R\mathbf{k}$ is a vector field in $\mathbb{R}^3$ where the first partial derivatives of P, Q, and R exist, then

$$\text{div } \mathbf{F} = \nabla \cdot \mathbf{F} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}$$

– Curl and divergence lead to another form of Green's Theorem.

- Let $C = \mathbf{r}(t)$ be a curve enclosing D satisfying the conditions of Green's Theorem.
- Recall $\mathbf{n}(t) = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|}$ is the unit normal vector of a curve C, then

$$\oint_C \mathbf{F} \cdot \mathbf{n}\, ds = \iint_D \text{div } \mathbf{F}(x, y)\, dA$$

The left side is read as "the line integral of the normal component of $\mathbf{F}$ along C."

§16.6 Parametric Surfaces and Their Areas

Good practice problems: 19-26, 33-36, 39-50 (+ review 3D coordinate systems for parameterizing surfaces)

– For $(u, v) \in D$ for some domain D, a parametric surface is defined by a vector function of two variables,

$$\mathbf{r}(u, v) = x(u, v)\mathbf{i} + y(u, v)\mathbf{j} + z(u, v)\mathbf{k} \tag{2}$$

– Parametric surfaces allow us to define a very general class of surfaces, as opposed to only the very specific surfaces we have seen in the past like spheres and cylinders.

– If a smooth parametric surface S is given by equation (2) for $(u, v) \in D$ and S is covered just once as (u, v) ranges through D, then the surface area of S is

$$A(S) = \iint_D |\mathbf{r}_u \times \mathbf{r}_v|\, dA$$

§16.7 Surface Integrals

Good practice problems: 5-32

– Surface integrals of **scalar functions** are calculated as

$$\iint_S f(x, y, z)\, dS = \iint_D f(\mathbf{r}(u, v))|\mathbf{r}_u \times \mathbf{r}_v|\, dA$$

where $\mathbf{r}(u, v)$ is the parameterization of the surface S.

– If we can write $z = g(x,y)$, this reduces to

$$\iint_S f(x,y,z)\,dS = \iint_D f(x,y,g(x,y))\sqrt{\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + 1}$$

– If it is possible to choose a unit normal vector $\mathbf{n}$ at every point (x,y,z) on a surface S (but not on the boundary) such that $\mathbf{n}$ varies continuously over a surface, then S is an **oriented surface**.

- $\mathbf{n}$ and $-\mathbf{n}$ offer two possible orientations of an oriented surface S.
- If S is closed (i.e. it is the boundary of a solid region E), the **positive orientation** is the one for which the normal vectors point outward from E.

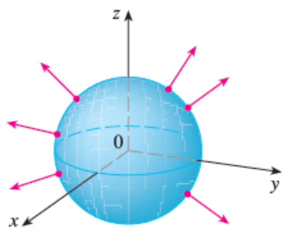


FIGURE 8
Positive orientation

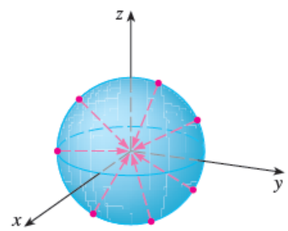


FIGURE 9
Negative orientation

– If $\mathbf{F}$ is a continuous vector field defined on an oriented surface S with unit normal vector $\mathbf{n}$ and $\mathbf{r}(u,v)$ is a parameterization of S, then the surface integral (or **flux integral**) of $\mathbf{F}$ over S is

$$\iint_S \mathbf{F}\cdot d\mathbf{S} = \iint_S \mathbf{F}\cdot\mathbf{n}\,dS = \iint_D \mathbf{F}\cdot(\mathbf{r}_u \times \mathbf{r}_v)\,dA$$

This is also known as the **flux** of $\mathbf{F}$ across S.

– If we can write $z = g(x,y)$, this reduces to

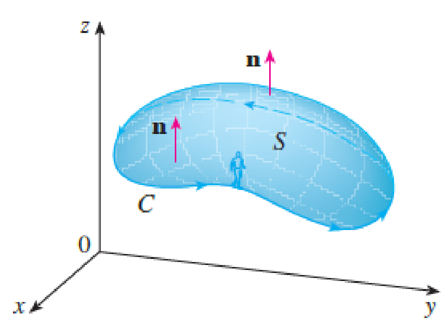
$$\iint_S \mathbf{F}\cdot d\mathbf{S} = \iint_D (-Pg_x - Qg_y + R)\,dA$$

where $\mathbf{F} = \langle P, Q, R\rangle$

§16.8 Stokes' Theorem

Good practice problems: 2-15

– If we have an oriented surface with unit normal vector $\mathbf{n}$:



the orientation of S induces a **positive orientation of the boundary curve** C, meaning that if you were to walk in a positive direction around C with your head pointing in the direction of $\mathbf{n}$, then the surface would be on your left.

- Let S be a piecewise-smooth oriented surface bounded by a simple, closed, piecewise-smooth curve C with positive orientation. If $\mathbf{F}$ is a vector field whose components have continuous partial derivatives on an open region in $\mathbb{R}^3$ containing S, then

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \iint_S \text{curl } \mathbf{F} \cdot d\mathbf{S}$$

- Note that we can apply the simplification from the last item in the notes for §16.7 where curl $\mathbf{F} = \langle P, Q, R \rangle$

- If S_1 and S_2 are oriented surfaces bounded by the same curve C and the conditions of Stokes' Theorem are satisfied, then

$$\iint_{S_1} \text{curl } \mathbf{F} \cdot d\mathbf{S} = \int_C \mathbf{F} \cdot d\mathbf{r} = \iint_{S_2} \text{curl } \mathbf{F} \cdot d\mathbf{S}$$

  This sometimes allows us to choose more convenient surfaces over which to take flux integrals.

§16.9 The Divergence Theorem

Good practice problems 1-15 (+ review triple integrals)

- Let E be a simple solid region bounded by a surface S with positive orientation. If $\mathbf{F}$ is a vector field whose components have continuous partial derivatives on an open region that contains S, then

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iiint_E \text{div } \mathbf{F} \, dV$$

Integral Types

	Scalar Function	Vector Field
Line Integral	$\int_C f \, ds = \int_a^b (\mathbf{r}(t)) \lVert \mathbf{r}'(t) \rVert \, dt$	$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_a^b \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t) \, dt$
Surface Integral	$\iint_S f \, dS = \iint_D f(\mathbf{r}(u,v)) \lVert \mathbf{r}_u \times \mathbf{r}_v \rVert \, dA$	$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iint_D \mathbf{F}(\mathbf{r}(u,v)) \cdot (\mathbf{r}_u \times \mathbf{r}_v) \, dA$