1 Highlights (§16.4 Green's Theorem)

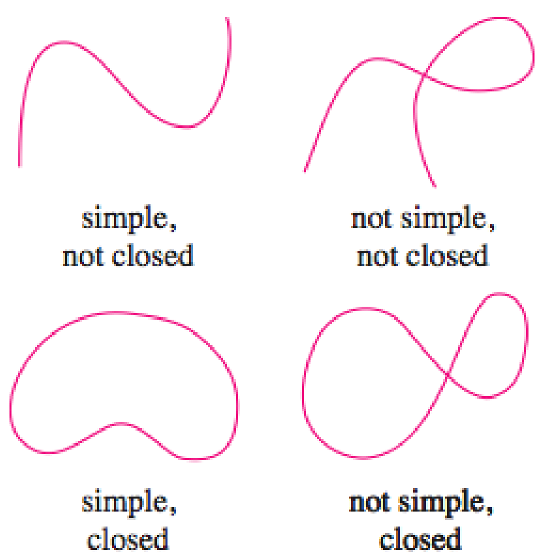
– Disclaimer: This is NOT a complete list of what you need to understand, additional material in the text may appear on tests. In particular, we had no lab on §16.3, but you need to understand it.

– Green's Theorem gives the relationship between line integrals around a simple closed curve C and a double integral over the plane region D which lies inside C.

– **Green's Theorem**: Let C be a positively oriented, piecewise-smooth, simple closed curve in the plane and let D be the region bounded by C. If P and Q have continuous partial derivatives on the open region that contains D, then

$$\int_C P\,dx + Q\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA \tag{1}$$

- $C = \partial D$ and $\int_C P\,dx + Q\,dy = \oint P\,dx + Q\,dy$ are alternate notations
- Recall that a smooth curve has no sharp corners or cusps; when its tangent vector turns, it does so continuously.
- Simple and closed curves:



– The above version of Green's Theorem requires D to be simple, but it can work on regions that are finite unions of simple regions and regions that are not simply-connected (i.e. containing holes) – see pages 1111-1113

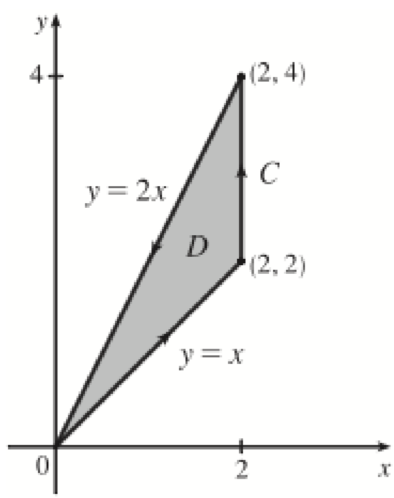
Problems

Example 1 (# 5): Use Green's Theorem to evaluate $\int_C xy^2\,dx + 2x^2y\,dy$ where C is the triangle with vertices (0,0), (2,2), and (2,4).

Solution. We can immediately apply Green's Theorem:

$$\begin{aligned}\int_C xy^2\,dx + 2x^2y\,dy &= \iint_D \left[\frac{\partial}{\partial x}(2x^2y) - \frac{\partial}{\partial y}(xy^2)\right]\,dA \\ &= \iint_D 4xy - 2xy\,dA = 2\iint_D xy\,dA\end{aligned}$$

In order to do the double integral here, we need to set bounds for x and y enclosing the triangle:



That is, $D = \{(x,y)\,|\,0 \leq x \leq 2, x \leq y \leq 2x\}$, which we can use the evaluate the double integral:

$$\begin{aligned}\int_C xy^2\,dx + 2x^2y\,dy &= 2\int_0^2 \int_x^{2x} xy\,dy\,dx\\ &= 2\int_0^2 \left[\frac{xy^2}{2}\right]_{y=x}^{y=2x} dx\\ &= \int_0^2 4x^3 - x^3\,dx = \frac{1}{2}\int_0^2 3x^3\,dx\\ &= \left[\frac{3x^4}{4}\right]_0^2 = \mathbf{12}\end{aligned}$$

Example 2: Use Green's Theorem to evaluate $\int_C \mathbf{F}\cdot d\mathbf{r}$ where $\mathbf{F}(x,y) = \langle e^x + x^2y, e^y - xy^2\rangle$ and C is the circle $x^2+y^2 = 25$ oriented counterclockwise.

Solution. Let D be the region enclosed by C.

$$\begin{aligned}\int_C \mathbf{F}\cdot d\mathbf{r} &= \int_C (e^x + x^2y)\,dx + (e^y - xy^2)\,dy && (\textstyle\int_C P\,dx + Q\,dy \text{ form})\\ &= \iint_D \left[\frac{\partial}{\partial x}(e^y - xy^2) - \frac{\partial}{\partial y}(e^x + x^2y)\right] dA && (\text{Green's Theorem})\\ &= \iint_D -y^2 - x^2\,dA = -\iint_D x^2+y^2\,dA && (\text{Differentiate})\\ &= -\int_0^{2\pi}\int_0^5 r(r^2)\,dr\,d\theta && (\text{Switch to polar coordinates})\\ &= -2\pi\left[\frac{r^4}{4}\right]_0^5 = -\frac{\mathbf{625}\boldsymbol{\pi}}{\mathbf{2}}\end{aligned}$$

Example 3 (# 13): Use Green's Theorem to evaluate $\int_C \mathbf{F}\cdot d\mathbf{r}$ where $\mathbf{F}(x,y) = \langle y - \cos y, x\sin y\rangle$ where C is the circle $(x-3)^2 + (x+4)^2 = 4$ oriented clockwise.

Solution. Let D be the region enclosed by C. Since Green's Theorem is defined for a positively oriented curve C, we will need to traverse C in reverse, which requires us to multiply by -1 so we will find $-\int_{-C} \mathbf{F}\cdot d\mathbf{r}$.

$$\begin{aligned}
\int_C \mathbf{F}\cdot d\mathbf{r} &= -\int_{-C} \mathbf{F}\cdot d\mathbf{r} && \text{(Use a positively oriented curve } -C\text{)}\\
&= -\int_{-C} (y-\cos y)\,dx + x\sin y\,dy && \left(\textstyle\int_{-C} P\,dx + Q\,dy \text{ form}\right)\\
&= -\iint_D \left[\frac{\partial}{\partial x}(x\sin y) - \frac{\partial}{\partial y}(y-\cos y)\right] dA && \text{(Green's Theorem)}\\
&= -\iint_D [\sin y - (1+\sin y)]\; dA && \text{(Differentiate)}\\
&= -\iint_D -1\,dA = \iint_D 1\,dA = \mathbf{4}\pi && \text{(The area of the circle)}
\end{aligned}$$

Example 4 (#28): Calculate $\int_C \mathbf{F}\cdot d\mathbf{r}$ where $\mathbf{F}(x,y) = \langle x^2+y, 3x-y^2\rangle$ and C is a simple positively oriented curve bounding a region D with area 6.

Solution. This problem is phrased a bit differently than the others. Here, we don't know much about C. In the problems above, Green's Theorem was mostly a quicker way to evaluate line integrals that we could otherwise solve, but we could not evaluate the line integral here via the techniques of §16.1 because we do not have enough information about C.

As always, be sure the problem fits the requirements for the result we wish to use – C is simple, closed, positively oriented, and the partial derivatives of the vector field $\mathbf{F}$ are continuous – so Green's theorem is applicable.

$$\begin{aligned}
\int_C \mathbf{F}\cdot d\mathbf{r} &= \int_C (x^2+y)\,dx + (3x-y^2)\,dy && \left(\textstyle\int_C P\,dx + Q\,dy \text{ form}\right)\\
&= \iint_D \left[\frac{\partial}{\partial x}(3x-y^2) - \frac{\partial}{\partial y}(x^2+y)\right] dA && \text{(Green's Theorem)}\\
&= \iint_D 3-1\,dA = 2\iint_D 1\,dA && \text{(Differentiate)}\\
&= 2A(D) = 2(6) = \mathbf{12} && \left(\textstyle\iint_D 1 \text{ is the area of } D\right)
\end{aligned}$$

Green's Theorem allowed us to evaluate a line integral of a vector field knowing only that C is a simple closed curve and the area it encloses, which we could not do previously.