Limits and Continuity

1 Intro

- In functions of 1 variable, a limit as $x \to a$ exists only if the left-sided and right-sided limits are equal.
- In functions of 2 variables, a limit as $(x, y) \to (a, b)$ exists only if limits along all smooth space curves $(x, y(x))$ approaching (a, b) are equal.
- There are infinitely many such curves, so we cannot simply show both one-sided limits are equal to find the limit as we could in functions of 1 variable.
- To show a limit does not exist, we can find 2 space curves along which the limits are not equal.
- To find the values of limits that do exist, we must use prior knowledge of a function's continuity, the δ-ε definition of the limit, or the Squeeze Theorem.
- All of these ideas scale up to functions of n variables.

2 Limits and Continuity

1 (Definition of a limit) Let f be a function of two variables whose domain D includes points arbitrarily close to (a, b). Then we say the limit of $f(x, y)$ as (x, y) approaches (a, b) is L and we write

$$\lim_{(x,y)\to(a,b)} f(x, y) = L$$

if for every number $\varepsilon > 0$, there exists a number $\delta_\varepsilon > 0$ such that for all (x, y) in the domain where $0 < \sqrt{(x-a)^2 + (y-b)^2} < \delta_\varepsilon$, we have $|f(x, y) - L| < \varepsilon$.

This says, intuitively, there exists a circle (of radius δ_ε) centered at (a, b) wherein $f(x, y)$ is within ε units of L no matter how small we make ε (except possibly at the point (a, b) itself).

Typically, δ depends on ε (and possibly x and y), which is emphasized with the δ_ε notation.

A function $f(x, y)$ is **continuous at the point** (a, b) if $\lim_{(x,y)\to(0,0)} f(x, y) = f(a, b)$. Some common functions are continuous:

- Polynomials are continuous everywhere

- Rational functions (a polynomial divided by a polynomial) are continuous on their domains.
  - $\frac{1}{x+y}$ is continuous for all (x,y) such that $y \neq -x$ (i.e. where the denominator is not 0).
- $\sin(x)$, $\cos(x)$, e^x are continuous everywhere, $\ln(x)$ is continuous for $x > 0$
- If f and g are continuous at (a,b), so are $f \pm g$, $f \cdot g$, $f \circ g$ (composition of functions)

3 The Squeeze Theorem

2 (The Squeeze Theorem) If $f(x,y) \leq g(x,y) \leq h(x,y)$ when (x,y) is near (a,b) (except possibly at (x,y)) and

$$\lim_{(x,y)\to(a,b)} f(x,y) = \lim_{(x,y)\to(a,b)} h(x,y) = L$$

then $\lim_{(x,y)\to(a,b)} g(x,y) = L$

We will find it useful to use absolute values $|f(x,y)|$ and $|h(x,y)|$ bounding the absolute value of our given function $|g(x,y)|$.

Commonly, if we think the limit is L, we can simply use $f(x,y) = L$ and find an upper bounding function $|h(x,y)|$ with the same limit.

4 Problems

Example 1 Find $\lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2}$ if it exists or show that it does not exist.

Let's try finding the limit along an arbitrary line $y = mx$ $(m \neq 0)$ through $(0,0)$,

$$\lim_{(x,mx)\to(0,0)} \frac{mx^2}{x^2+m^2x^2} = \lim_{(x,mx)\to(0,0)} \frac{m}{1+m^2} = \frac{m}{1+m^2}$$

$y = x$ and $y = 2x$ are valid forms of the $y = mx$ we plugged in, but using $m = 1$ and $m = 2$ produce **different** values for limits along different space curves, so the limit **does not exist**.

(By choosing an arbitrary line $y = mx$, we only had to take one limit as opposed to trying once with $y = x$ and another time with $y = 2x$. Since finding the limit along $y = mx$ is typically no more difficult than using $y = x$, this is an effective practice.)

Example 2 Find $\lim_{(x,y)\to(2,1)} 2xy + 5x^4 - x^2y^6$ if it exists or show that it does not exist.

Since this function is a polynomial, it is continuous everywhere, so we can just plug in $x = 2, y = 1$,

$$\lim_{(x,y)\to(2,1)} 2xy + 5x^4 - x^2y^6 = 2(2)(1) + 5(2)^4 - (2)^2(1)^6 = \mathbf{80}$$

Example 3 Find $\lim\limits_{(x,y,z)\to(3,0,1)} 2e^{-xy}\sin\left(\frac{\pi z}{2}\right)$ if it exists or show that it does not exist.

We know that $f(x) = 2e^x$ and $g(x) = -xy$ are continuous, so the composition $2e^{-xy}$ is continuous. Similarly, $\sin(x)$ and $\frac{\pi x}{2}$ are continuous, so the composition $\sin\left(\frac{\pi z}{2}\right)$ is continuous. The product of continuous functions is continuous, so the whole function $2e^{-xy}\sin\left(\frac{\pi z}{2}\right)$ is continuous, then

$$\lim_{(x,y,z)\to(3,0,1)} 2e^{-xy}\sin\left(\frac{\pi z}{2}\right) = 2e^{3\cdot 0}\sin\left(\frac{\pi}{2}\right) = 2\cdot 1\cdot 1 = \mathbf{2}$$

Example 4 (Squeeze Theorem 1) Find $\lim\limits_{(x,y)\to(0,0)} \frac{y^2(1-\cos(2x))}{x^4+y^2}$ if it exists or show that it does not exist.

It appears we can cancel the terms in the denominator by finding the limit along $y = mx^2$

$$\begin{aligned}\lim_{(x,mx^2)\to(0,0)} \frac{m^2x^4(1-\cos(2x))}{x^4+m^2x^4} &= \lim_{(x,mx^2)\to(0,0)} \frac{m^2(1-\cos(2x))}{1+m^2} \\ &= \frac{m^2(1-1)}{1+m^2} = 0\end{aligned}$$

Solving along a line $y = mx$ appears to maybe be useful as well:

$$\begin{aligned}\lim_{(x,x)\to(0,0)} \frac{m^2x^2(1-\cos(2x))}{x^2(x^2+m^2)} &= \lim_{(x,x)\to(0,0)} \frac{m^2(1-\cos(2x))}{x^2+m^2} \\ &= \frac{m^2(1-1)}{0+m^2} = 0\end{aligned}$$

Since all of these produce the same value, it seems maybe the limit actually exists as 0, but we need to prove it, so we will attempt to use the Squeeze Theorem. Since the limit is 0, we can use simply the constant function 0 as our lower bound,

$$0 \le \left|\frac{y^2(1-\cos(2x))}{x^4+y^2}\right| \le |h(x,y)|$$

So we need a function h whose absolute value is larger than the absolute value of our function, but also has a limit of 0 as $(x,y)\to(0,0)$. We notice

$$\left|\frac{y^2(1-\cos(2x))}{x^4+y^2}\right| = \frac{y^2}{x^4+y^2}\,|1-\cos(2x)| \le |1-\cos(2x)|$$

Since $0 \le \frac{y^2}{x^4+y^2} \le 1$, and $\lim\limits_{(x,y)\to(0,0)} 1-\cos(2x) = 0$, then $\lim\limits_{(x,y)\to(0,0)} \frac{x^4(1-\cos(2x))}{x^4+y^2} = 0$ by the Squeeze Theorem.

Example 5 (δ-ε and Squeeze Theorem 2) Find $\lim\limits_{(x,y)\to(0,0)} \frac{2x^2y}{x^2+y^2}$ if it exists or show that it does not exist.

Along a line $y = mx$,

$$\lim_{(x,mx)\to(0,0)} \frac{2mx^3}{x^2 + m^2x^2} = \lim_{(x,mx)\to(0,0)} \frac{2mx}{1+m^2} = 0$$

So we know the limit is either 0 or does not exist. No other simple curves seem to lead to nonzero limits, so let's attempt prove the limit is 0.

Method 1: Definition of a Limit

To show that 0 satisfies the definition of the limit, we need to show that for a given, arbitrary $\varepsilon > 0$, we can find a $\delta_\varepsilon > 0$ such that if $\sqrt{x^2+y^2} < \delta_\varepsilon$, then $\left|\frac{2x^2y}{x^2+y^2} - 0\right| < \varepsilon$.

Fix the value of ε and then we will try to reduce the latter absolute value. Since $x^2 \le x^2+y^2$ and each is positive,

$$\left|\frac{2x^2y}{x^2+y^2} - 0\right| = 2|y|\frac{x^2}{x^2+y^2} \le 2|y| \le 2\sqrt{x^2+y^2} < 2\delta_\varepsilon$$

Choose $\delta_\varepsilon = \frac{\varepsilon}{2}$. Then if we let $\sqrt{x^2+y^2} < \delta_\varepsilon$ and use the above, for the given ε,

$$\left|\frac{2x^2y}{x^2+y^2} - 0\right| < 2\delta(x,y,\varepsilon) = 2\frac{\varepsilon}{2} = \varepsilon$$

Thus, the limit is 0 by the definition of a limit.

Method 2: Squeeze Theorem As shown above,

$$0 \le \left|\frac{2x^2y}{x^2+y^2}\right| = \frac{2x^2}{x^2+y^2}|y| \le 2|y|$$

The limit of $2|y|$ as $(x,y) \to (0,0)$ is 0, so $\lim\limits_{(x,y)\to(0,0)} \frac{2x^2y}{x^2+y^2} = 0$ by the Squeeze Tpheorem.

Example 6 (Squeeze Theorem 3) Find $\lim\limits_{(x,y)\to(0,0)} \frac{xy}{\sqrt{x^2+y^2}}$ if it exists or show that it does not exist.

Finding the limit along the line $y = mx$,

$$\begin{aligned}\lim_{(x,mx)\to(0,0)} \frac{mx^2}{\sqrt{x^2+m^2x^2}} &= \lim_{(x,mx)\to(0,0)} \frac{mx^2}{x\sqrt{1+m^2}} \\ &= \lim_{(x,mx)\to(0,0)} \frac{mx}{\sqrt{1+m^2}} = 0\end{aligned}$$

Finding the limit along a parabola $y = mx^2$,

$$\begin{aligned}\lim_{(x,mx^2)\to(0,0)} \frac{mx^3}{\sqrt{x^2+m^2x^4}} &= \lim_{(x,mx^2)\to(0,0)} \frac{mx^3}{\sqrt{x^2(1+m^2x^2)}} \\ &= \lim_{(x,mx^2)\to(0,0)} \frac{mx^3}{x\sqrt{1+m^2x^2}} \\ &= \lim_{(x,mx^2)\to(0,0)} \frac{mx^2}{\sqrt{1+m^2x^2}} = \frac{0}{\sqrt{1+0}} = 0\end{aligned}$$

We can see that similarly, $x = my^2$ would yield 0 as well, so it seems perhaps the limit exists, so let's try the Squeeze Theorem:

$$0 \leq \left| \frac{xy}{\sqrt{x^2+y^2}} \right| = |x| \frac{|y|}{\sqrt{x^2+y^2}} = |x| \frac{\sqrt{y^2}}{\sqrt{x^2+y^2}} \leq |x|$$

Since the limit of $|x|$ for $(x, y) \to (0, 0)$ is 0, $\lim\limits_{(x,y)\to(0,0)} \frac{xy}{\sqrt{x^2+y^2}} = 0$ by the Squeeze Theorem.

Example 7 Determine the set of points at which the function is continuous.

$$f(x) = \begin{cases} \frac{x^2y^3}{2x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ 1 & \text{if } (x,y) = (0,0) \end{cases}$$

First, notice that f is a rational function with a domain consisting of all points except $(0, 0)$, so it must be continuous on its domain, but what about at the point $(0, 0)$ itself?

It will be continuous at $(0, 0)$ only if $\lim\limits_{(x,y)\to(0,0)} f(x, y) = f(0, 0) = 1$, so we need to find the limit or show that it does not exist. Notice that along the line $y = x$, we find

$$\lim_{(x,x)\to(0,0)} \frac{x^5}{3x^2} = \lim_{(x,x)\to(0,0)} \frac{x^3}{3} = \frac{0}{3} = 0$$

Therefore, the limit as $(x, y) \to (0, 0)$ is 0 or does not exist – in either case, the limit is not $f(0, 0)$, so continuity fails. In set notation, f is continuous on $\big\{(x, y) \in \mathbb{R}^2 \mid (x, y) \neq (0, 0)\big\}$.

5 General Strategy for Finding $\lim\limits_{(x,y)\to(a,b)} f(x, y)$

1. Do we already know f is continuous near (a, b)? (Is f constructed from $\sin(x)$, $\cos(x)$, e^x, polynomial, etc, defined near (a, b)? If so, we immediately have a limit $f(a, b)$.
2. If not, find the limit along convenient smooth space curves, such as straight lines along the lines $y = b$, $x = a$, or better yet, an arbitrary line through (a, b) with slope m:

$$\begin{aligned} y - b &= m(x - a) \\ y &= m(x - a) + b \end{aligned}$$

   (**Convenient** space curves often allow you to cancel troublesome terms from the denominator.)
3. Note that many of the limits you will see will be as $(x, y) \to (0, 0)$, which makes these formulas simpler (for example, $y = mx$ instead of $y = m(x - a) + b$).
4. If the limit along one of these curves does not exist or two such limits do not match (for example, if the value changes for different values of m), the limit of interest will not exist.
5. Otherwise, try some other curves, such as $y = (x - a)^n + b$ or $x = (y - b)^n + a$.
6. If these simple curves produce matching limits, we may suspect the limit exists, and we should undertake another strategy to prove the limit is what we think it is, such as a δ-ε proof or the Squeeze Theorem.