1 Highlights (§15.4)

- Disclaimer: This is NOT a complete list of what you need to understand. It is very necessary to do practice problems.
- Cartesian to polar coordinates: $r^2 = x^2 + y^2$, $x = r\cos\theta$, $y = r\sin\theta$.
- If f is continuous on $D = \{(r,\theta)\,|\,\alpha \le \theta \le \beta, h_1(\theta) \le r \le h_2(\theta)\}$, then

$$\iint_D f(x,y)\,dA = \int_\alpha^\beta \int_{h_1(\theta)}^{h_2(\theta)} f(r\cos\theta, r\sin\theta) r\,dr\,d\theta$$

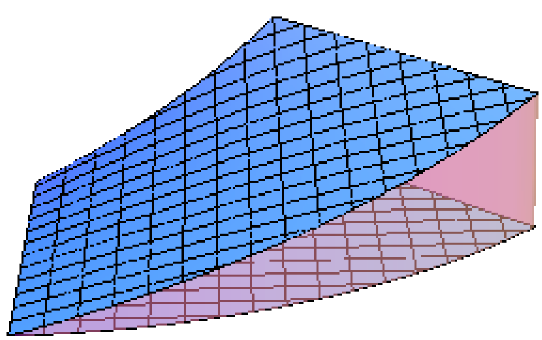
- It is very helpful in this section to recall some trigonometric identities.
- Draw the domains and/or figures if at all possible! Useful symmetries can make some problems much shorter and easier.

2 Problems

Example 1 (#13): Evaluate the integral by changing to polar coordinates $\iint_D \arctan\left(\frac{y}{x}\right)\,dA$ where $D = \{(x,y)\,|\,1 \le x^2+y^2 \le 4, 0 \le y \le x\}$.

Since $1 \le x^2 + y^2 \le 4$, we can plug in $r^2 = x^2 + y^2$ to find $1 \le r \le 2$.

Next, we need to find bounds for θ. Since $x \ge y \ge 0$, we are be limited to the first quadrant where $y \le x$, so we must have the region below $y = x$ in the first quadrant, which makes a 45 deg, or $\frac{\pi}{4}$, so $0 \le \theta \le \frac{\pi}{4}$



We can now set up the integral:

$$\begin{aligned}\iint_D \arctan\left(\frac{y}{x}\right)\,dA &= \int_0^{\frac{\pi}{4}}\int_1^2 r\arctan\left(\frac{r\sin\theta}{r\cos\theta}\right)\,dr\,d\theta \\ &= \int_0^{\frac{\pi}{4}}\int_1^2 r\arctan\left(\tan\theta\right)\,dr\,d\theta = \int_0^{\frac{\pi}{4}}\int_1^2 r\theta\,dr\,d\theta \\ &= \int_0^{\frac{\pi}{4}}\theta\,d\theta\int_1^2 r\,dr = \left[\frac{\theta^2}{2}\right]_0^{\frac{\pi}{4}}\left[\frac{r^2}{2}\right]_1^2 = \frac{\pi^2}{32}\cdot\frac{3}{2} = \frac{3\pi^2}{64}\end{aligned}$$

Note that splitting the integral in the last line was only possible because the limits of integration were constants and we could write the integrand as a product of functions of r and θ exclusively.

Also note $\arctan(\tan\theta) = \theta$ is not true everywhere, but it is true on $0 \leq \theta \leq \frac{\pi}{2}$.

Example 2 Use a double integral to find the area of the region enclosed by both circles $(x-1)^2 + y^2 = 1$ and $x^2 + (y-1)^2 = 1$.

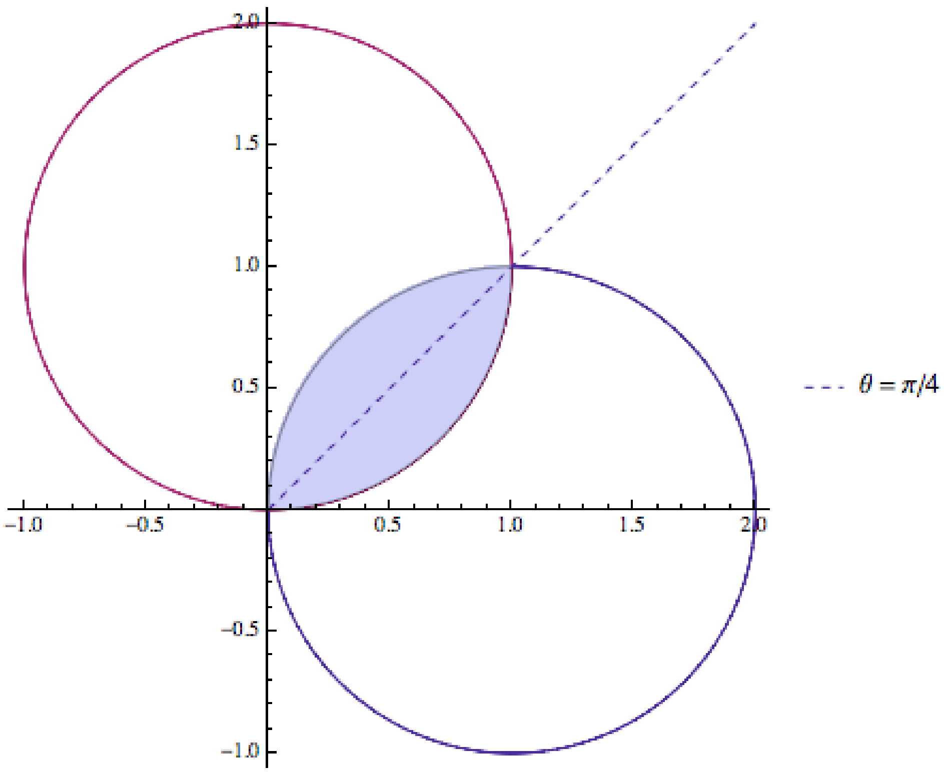
We have been using double integrals for volume, but if we set the height of the figure to 1 unit within the region, we will get the area.

We notice the region will lie in the first quadrant between angles 0 and $\frac{\pi}{2}$, but the part from 0 to $\frac{\pi}{4}$ will be exactly half the area, so we can integrate θ in this region and multiply by 2.

Notice the radius in this region will go from the origin to the upper circle $x^2 + (y-1)^2 = 1$, so we can solve for r (we used the wrong circle in lab! This would have been okay if we adjusted the bounds on theta to $\pi/4 \leq \theta \leq \pi/2$):

$$\begin{aligned} x^2 + (y-1)^2 &= 1 \\ x^2 + y^2 - 2y + 1 &= 1 \\ r^2 - 2r\sin\theta &= 0 \\ r(r - 2\sin\theta) &= 0 \end{aligned}$$

Then we have $0 \leq r \leq 2\sin\theta$ where $0 \leq \theta \leq \frac{\pi}{4}$:



Then the area is

$$A = \iint_D 1 \cdot \, dA = 2\int_0^{\frac{\pi}{4}} \int_0^{\sin\theta} r \, dr \, d\theta = 2\int_0^{\frac{\pi}{4}} \left[\frac{r^2}{2}\right]_0^{\sin\theta} d\theta$$

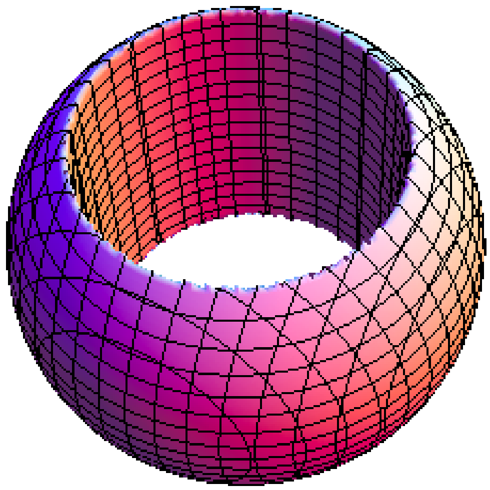
$$= \int_0^{\frac{\pi}{4}} \sin^2\theta \, d\theta = \frac{1}{2}\int_0^{\frac{\pi}{4}} (1 - \cos 2\theta) \, d\theta$$

$$= \frac{1}{2}\left[\theta + \frac{\sin 2\theta}{2}\right]_0^{\frac{\pi}{4}} = \frac{1}{2}\left[\frac{\pi}{4} + \frac{1}{2}\right] = \frac{1}{8}(\pi + 2)$$

where we used the half-angle formula in the next to last line.

Example 3 (The Napkin Ring Problem, #20): A cylindrical drill with radius r_1 is used to bore a hole through the center of a sphere with radius r_2. Find the volume of the ring-shaped solid that remains and write it in terms of the height of the hole.

We can center the sphere at $(0, 0, 0)$ if we like and it makes sense to find the volume above the xy-plane and multiply by 2, assuming the cylindrical hole is perpendicular to the xy-plane, so we need the distance between a point (r, θ) on the xy-plane and the half sphere directly above.



To find the height, we draw a right triangle between $(0, 0)$, (r, θ), and the sphere directly above (r, θ), then we know the lower leg is of length r and the hypotenuse is r_2, then the height is $\sqrt{r_2^2 - r^2}$ by the Pythagorean Theorem.

Clearly, our region will be where $r_1 \leq r \leq r_2$ and $0 \leq \theta \leq 2\pi$, then we have

$$\begin{aligned} V &= 2\int_0^{2\pi}\int_{r_1}^{r_2} r\sqrt{r_2^2 - r^2}\,dr\,d\theta = 2\int_0^{2\pi} -\frac{1}{2}\int_{r_2^2-r_1^2}^{0} \sqrt{u}\,du\,d\theta \\ &= \int_0^{2\pi}\int_0^{r_2^2-r_1^2} \sqrt{u}\,du\,d\theta = \int_0^{2\pi}\left[\frac{2}{3}u^{\frac{3}{2}}\right]_0^{r_2^2-r_1^2} d\theta \\ &= \frac{2}{3}\left(r_2^2 - r_1^2\right)^{\frac{3}{2}}\int_0^{2\pi} d\theta = \frac{4\pi}{3}\left(r_2^2 - r_1^2\right)^{\frac{3}{2}} \end{aligned}$$

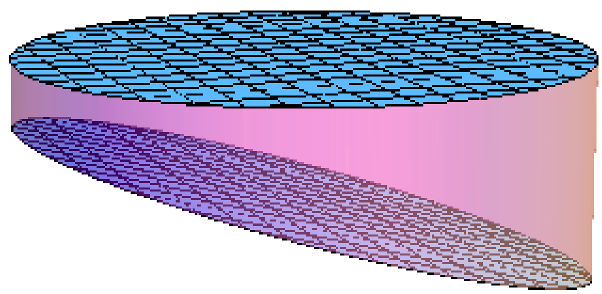
This gives us the volume, but if we wish to write it in terms of the height of the hole, we can find the height at $r = r_1$, which is $\sqrt{r_2^2 - r_1^2}$ for the top half, and so $h = 2\sqrt{r_2^2 - r_1^2}$, then the volume can be written

$$V = \frac{4\pi}{3}\left(\sqrt{r_2^2 - r_1^2}\right)^3 = \frac{\pi}{3}\frac{1}{2}h^3 = \frac{\pi h^3}{6}$$

Example 4 (#35): A swimming pool is circular with 40-ft diameter. The depth is constant along east-west lines and increases linearly from 2 ft at the sound end to 7 ft at the north end. Find the volume of water in the pool.

If we center the pool at (0,0,0) with the x-axis representing the east-west direction, then we know the domain is where $x^2+y^2 \leq 400$, or $\{(r,\theta)\,|\,0 \leq r \leq 20, 0 \leq \theta \leq 2\pi\}$ in polar coordinates.

The depth of the pool will be independent of x since the depth is constant along east-west lines, then the depth function $f(x,y)$ will start at $f(0,-20) = 2$ in the shallow end and $f(0,20) = 7$ in the deep end, with a line connecting the two.

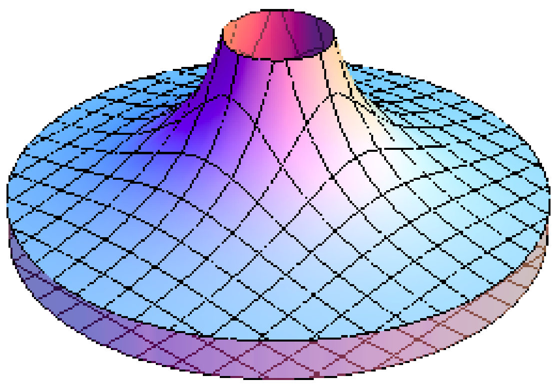


Thus, for a line $f(x,y) = my + b$, we can find $m(20-(-20)) = 7-2$, or $m = \frac{1}{8}$, and $b = f(0,0) = \frac{9}{2}$, so we have $f(x,y) = \frac{1}{8}y + \frac{9}{2}$. Evaluating the volume via polar coordinates:

$$\begin{aligned} V &= \iint_D f(x,y)dA = \int_0^{2\pi}\int_0^{20} \frac{1}{8}r^2\sin\theta + \frac{9}{2}r\,dr\,d\theta \\ &= \int_0^{2\pi} \frac{\sin\theta}{8}\frac{8000}{3} + \frac{9}{2}\frac{400}{2}\,d\theta = \int_0^{2\pi} \frac{1000}{3}\sin\theta + 900\,d\theta \\ &= \frac{1000}{3}\left[-\cos\theta\right]_0^{2\pi} + \left[900\theta\right]_0^2 0 = 1800\pi \text{ ft}^3 \end{aligned}$$

Example 5 (#37): Find the average value of the function $f(x,y) = \frac{1}{\sqrt{x^2+y^2}}$ on the annulus $a^2 \leq x^2 + y^2 \leq b^2$ (i.e. between circles of radii a and b).

To find the average value, we first need the area of the region in the xy-plane, i.e. the area of the outer circle minus the area of the inner circle: $\pi(b^2 - a^2)$.



Our region in polar coordinates will be $D = \{(r, \theta) \,|\, a \leq r \leq b, 0 \leq \theta \leq 2\pi\}$, and our function in polar coordinates is $f(r, \theta) = \frac{1}{r}$, then we have

$$\begin{aligned} f_{avg} &= \frac{1}{A(D)} \int_0^{2\pi} \int_a^b \frac{1}{r} r \, dr \, d\theta = \frac{1}{\pi(b^2 - a^2)} \int_0^{2\pi} \int_a^b dr \, d\theta \\ &= \frac{b-a}{\pi(b^2 - a^2)} \int_0^{2\pi} d\theta = \frac{2\pi(b-a)}{\pi(b^2 - a^2)} = \frac{2(b-a)}{(b-a)(b+a)} = \frac{2}{a+b} \end{aligned}$$