§12.1 3D Coordinates

Example 1 What does the equation $(x-3)^2+(y+2)^2+z^2=4$ represent in 3D space? What about $1 \leq (x-3)^2+(y+2)^2+z^2 \leq 4$?

Solution: The first formula is for a sphere of radius $\sqrt{4}=2$, centered at $(3,-2,0)$ (not filled in).

The second is all space between the sphere centered at the same point with radius 1 and the sphere centered at the same point with radius 2.

§12.2-4 Vectors and Products

Example 1 What is the angle between vectors $\mathbf{i}+3\mathbf{j}-4\mathbf{k}$ and $-\mathbf{i}-\mathbf{j}+4\mathbf{k}$?

Solution:

$$\begin{aligned}
\cos(\theta) &= \frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}||\mathbf{b}|} \\
&= \frac{(1\cdot(-1))+(3\cdot(-1))+((-4)\cdot 4)}{\sqrt{1^2+3^2+(-4)^2}\sqrt{(-1)^2+(-1)^2+4^2}} \\
&= \frac{-20}{\sqrt{26}\sqrt{18}} \\
\theta &= \arccos\left(\frac{-20}{\sqrt{26}\sqrt{18}}\right) \approx 2.75 \text{ radians} \approx 157.6^\circ
\end{aligned}$$

Example 2 Show that the vector $\text{orth}_{\mathbf{a}}\mathbf{b} = \mathbf{b} - \text{proj}_{\mathbf{a}}\mathbf{b}$ is orthogonal to $\mathbf{a}$. (It is called the orthogonal projection of $\mathbf{b}$.)

Solution:

$$\begin{aligned}
(\text{orth}_{\mathbf{a}}\mathbf{b})\cdot\mathbf{a} &= (\mathbf{b}-\text{proj}_{\mathbf{a}}\mathbf{b})\cdot\mathbf{a} && \text{by definition of orth. projection} \\
&= \mathbf{b}\cdot\mathbf{a} - \left(\frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}|^2}\mathbf{a}\right)\cdot\mathbf{a} && \text{by def'n of proj and distributive property} \\
&= \mathbf{b}\cdot\mathbf{a} - \left(\frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}|^2}\right)\mathbf{a}\cdot\mathbf{a} && \text{property 4 of the dot product} \\
&= \mathbf{b}\cdot\mathbf{a} - \left(\frac{\mathbf{a}\cdot\mathbf{b}}{\cancel{|\mathbf{a}|^2}}\right)\cancel{|\mathbf{a}|^2} && \text{property 1 of the dot product} \\
&= \mathbf{b}\cdot\mathbf{a} - \mathbf{b}\cdot\mathbf{a} && \text{property 2 of the dot product} \\
&= 0
\end{aligned}$$

The dot product of $\mathbf{a}$ and $\text{orth}_{\mathbf{a}}\mathbf{b}$ is zero, so they are orthogonal.

Example 3 Use a scalar projection to show that the distance from a point $P_1(x_1, x_2)$ to the line $ax + by + c = 0$ is

$$\frac{|ax_1 + bx_2 + c|}{\sqrt{a^2 + b^2}}$$

and find the distance from the point $(-2, 3)$ to the line $3x - 4y + 5 = 0$.

Solution: First, notice that the vector $\mathbf{n} = \langle a, b\rangle$ is perpendicular to the line, which we will show. Let $Q_1(a_1, b_1)$ and $Q_2(a_2, b_2)$ be points on the line, then we have

$$\begin{aligned} \mathbf{n} \cdot \overrightarrow{Q_1Q_2} &= a(a_2 - a_1) + b(b_2 - b_1) \\ &= (aa_1 + bb_2) - (aa_2 + bb_2) \\ &= -c - (-c) = 0 && \text{Since } Q_1 \text{ and } Q_2 \text{ are on the line} \end{aligned}$$

Since $\overrightarrow{Q_1Q_2}$ is in the direction of the line and this dot product is 0, we now see that $\mathbf{n}$ is perpendicular to the line.

Now we can see that the vector $\overrightarrow{P_1Q_2}$ will connect our point P_1 to a point on the line, Q_2, but it is not necessarily the shortest vector between the two. The shortest vector will be in the direction of n, since it is perpendicular to the line.

In order to find the length of the shortest vector connecting P_1 to the line will be $|\text{comp}_{\mathbf{n}}\overrightarrow{P_1Q_2}|$:

$$\begin{aligned} |\text{comp}_{\mathbf{n}}\overrightarrow{P_1Q_2}| &= \frac{|\mathbf{n} \cdot \overrightarrow{P_1Q_2}|}{|\mathbf{n}|} \\ &= \frac{|a(a_2 - x_1) + b(b_2 - x_2)|}{\sqrt{a^2 + b^2}} \\ &= \frac{|(aa_2 + bb_2) - (ax_1 + bx_2)|}{\sqrt{a^2 + b^2}} \\ &= \frac{|-c - (ax_1 + bx_2)|}{\sqrt{a^2 + b^2}} && \text{Since } Q_2 \text{ is on the line} \\ &= \frac{|ax_1 + bx_2 + c|}{\sqrt{a^2 + b^2}} && \text{Multiplying the inside by } -1 \text{ does} \\ & && \text{not change the absolute value} \end{aligned}$$

This is the desired result. Applying the formula to allows us to find the distance from $(-2, 3)$ to the line $3x - 4y + 5 = 0$:

$$\begin{aligned}\text{distance} &= \frac{|3(-2) - 4(3) + 5)}{\sqrt{3^2 + 4^2}} \\ &= \frac{|-13|}{5} = \frac{13}{5}\end{aligned}$$

Example 4 Find a unit vector that is orthogonal to both $\langle 0, 1, 3\rangle$ and $\langle -1, 0, 3\rangle$

Solution: We know the cross product of two vectors produces a vector orthogonal to both vectors, so we can find

$$\begin{aligned}\mathbf{v} &= \mathbf{v_1} \times \mathbf{v_2} \\ &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 1 & 3 \\ -1 & 0 & 3 \end{vmatrix} \\ &= \mathbf{i}(3-0) - \mathbf{j}(0+3) + \mathbf{k}(0+1) \\ &= \langle 3, -3, 1\rangle\end{aligned}$$

So we found a vector orthogonal to both of the given vectors, but we need to find a unit vector. A vector of different magnitude in the same direction as $\mathbf{v}$ will also be orthogonal, so we can find the unit vector in this direction by dividing $\mathbf{v}$ by its magnitude:

$$\begin{aligned}\mathbf{u} &= \frac{\mathbf{v}}{|\mathbf{v}|} \\ &= \frac{\langle 3, -3, 1\rangle}{\sqrt{3^2 + (-3)^2 + 1^2}} \\ &= \left\langle \frac{3}{\sqrt{19}}, -\frac{3}{\sqrt{19}}, \frac{1}{\sqrt{19}} \right\rangle\end{aligned}$$

§12.5 Equations of Lines and Planes

Example 1 Find parametric equations for the line through the point $(0, 1, 2)$ that is parallel to the plane $x + y + z = 2$ and perpendicular to the line $x = 1 + t$, $y = 1 - t$, $z = 2t$.

Solution: We know that the line will be perpendicular to both the plane's normal vector, $\langle 1, 1, 1\rangle$, and the direction vector of the given line, $\langle 1, -1, 2\rangle$.

To find a vector perpendicular to both of these, we find the cross product:

$$\begin{aligned}
\mathbf{v} &= \mathbf{v_1} \times \mathbf{v_2} \\
&= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & 1 \\ 1 & -1 & 2 \end{vmatrix} \\
&= \mathbf{i}(2+1) - \mathbf{j}(2-1) + \mathbf{k}(-1-1) \\
&= \langle 3, -1, -2 \rangle
\end{aligned}$$

Since this vector $\mathbf{v}$ is perpendicular to both of these, it will be the direction vector of our line. We also know the point $(0, 1, 2)$ is on our line, so we can find the parametric equations of the line as follows:

$$\langle x(t), y(t), z(t) \rangle = \langle 3t, 1-t, 2-2t \rangle$$

§12.6 Cylinders and Quadric Surfaces

Example 1 Reduce the equation $x^2 - y^2 + z^2 - 4x - 2y - 2z + 4 = 0$ to one of its standard forms and classify the surface.

Solution: First, we want to complete the square for each x, y, z by adding and subtracting convenient values as follows:

$$\begin{aligned}
x^2 - y^2 + z^2 - 4x - 2y - 2z + 4 &= 0 \\
(x^2 - 4x) - (y^2 + 2y) + (z^2 - 2z) + 4 &= 0 \\
(x^2 - 4x + 4) - 4 - (y^2 + 2y + 1) + 1 + (z^2 - 2z + 1) - 1 + 4 &= 0 \\
(x^2 - 4x + 4) - (y^2 + 2y + 1) + (z^2 - 2z + 1) &= 0 \\
(x-2)^2 - (y+1)^2 + (z-1)^2 &= 0 \\
(x-2)^2 + (z-1)^2 &= (y+1)^2
\end{aligned}$$

Using the chart of types of surfaces from §12.6, we see this as a horizontal cone with center $(2, -1, 1)$ and axis the horizontal line $\langle 2, t, 1 \rangle$

Example 2 Reduce the equation $4x^2 + y^2 + 4z^2 - 4y - 24z + 36 = 0$ to one of its standard forms and classify the surface.

Solution: Again, we want to complete the squares for y, z, but not x because we don't have a term in the form cx:

$$\begin{aligned}
4x^2 + y^2 + 4z^2 - 4y - 24z + 36 &= 0 \\
4x^2 + (y^2 - 4y) + (4z^2 - 24z) + 36 &= 0 \\
4x^2 + (y^2 - 4y + 4) - 4 + 4(z^2 - 6) + 36 &= 0 \\
4x^2 + (y-2)^2 + 4(z^2 - 6 + 9) - 36 + 32 &= 0 \\
4x^2 + (y-2)^2 + 4(z-3)^2 &= 4 \\
x^2 + \frac{(y-2)^2}{4} + (z-3)^2 &= 1
\end{aligned}$$

This formula is now in the form of an ellipsoid with center $(0, 2, 3)$, which will be elongated in the direction of the y axis.

§13.1 Vector Functions and Space Curves

Example 1 Let $\mathbf{r}(t) = \langle t\sin(t), t\cos(t), t+1\rangle$. Find $\lim_{t\to 0} \mathbf{r}(t)$ and $\lim_{t\to 6\pi} \mathbf{r}(t)$ for $0 \le t \le 6\pi$.

Solution: In order to apply a limit, we just apply it separately to each component of the space curve $\mathbf{r}(t)$:

$$\begin{aligned}
&\lim_{t\to 0} \mathbf{r}(t) = \langle 0, 0, 1\rangle \\
&\lim_{t\to 6\pi} \mathbf{r}(t) = \langle 0, 6\pi, 1 + 6\pi\rangle
\end{aligned}$$

Since $x(t) = t\sin(t), y(t) = t\cos(t)$, x and y spiral outward, getting larger and larger as t increases. $z(t) = t + 1$, so the plot moves upward steadily as t increases, beginning at $z = 1$.

§13.2 Derivatives and Integrals of Vector Functions

Example 1 Evaluate the integral $\int\langle\sin t, \cos t, t\rangle dt$

Solution: We can apply the integral to each component separately, so we have

$$\begin{aligned}
\int\langle\sin t, \cos t, t\rangle dt &= \left\langle \int \sin t dt, \int \cos t dt, \int t dt \right\rangle \\
&= \left\langle -\cos t, \sin t, \frac{t^2}{2} \right\rangle
\end{aligned}$$

§13.3 Arc Length and Curvature

Example 1: Find the vectors $\mathbf{T}(t)$, $\mathbf{N}(t)$, $\mathbf{B}(t)$, and the curvature $\kappa(t)$ of the space curve $\mathbf{r}(t) = \langle t^2, \frac{2}{3}t^3, t\rangle$ at point $\left(1, \frac{2}{3}, 1\right)$. What are these values called? Find the arc length from $t = 0$ to $t = 1$. What are these values geometrically?

Solution: Firstly, what is the t value such that $\mathbf{r}(t) = \left(1, \frac{2}{3}, 1\right)$? We can see that $t = 1$ is one such t value.

$\mathbf{T}(1)$ is the unit tangent vector to $\mathbf{r}(t)$ at $t = 1$, which we find as follows:

$$\begin{aligned}\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|} &= \frac{\langle 2t, 2t^2, 1\rangle}{\sqrt{4t^2 + 4t^4 + 1}}\\ &= \frac{\langle 2t, 2t^2, 1\rangle}{\sqrt{(2t^2+1)^2}}\\ &= \frac{\langle 2t, 2t^2, 1\rangle}{2t^2+1}\end{aligned}$$

Therefore, the unit tangent vector at the given point is $\mathbf{T}(1) = \langle \frac{2}{3}, \frac{2}{3}, \frac{1}{3}\rangle$

Next, we seek the principle unit normal vector $\mathbf{N}(1)$, which indicates the direction in which $\mathbf{r}(t)$ is turning at $t = 1$, which we find as follows:

$$\mathbf{N}(t) = \frac{\mathbf{T}'(t)}{|\mathbf{T}'(t)|}$$

where

$$\begin{aligned}\mathbf{T}'(t) &= \frac{\langle (2t^2+1)(2) - (2t)(4t), (2t^2+1)(4t) - (2t^2)(4t), (2t^2+1)(0) - (1)(4t)\rangle}{(2t^2+1)^2}\\ &= \frac{\langle 2 - 4t^2, 4t, -4t\rangle}{(2t^2+1)^2} = \frac{2}{(2t^2+1)^2}\langle 1 - 2t^2, 2t, -2t\rangle\end{aligned}$$

$$
\begin{aligned}
|\mathbf{T}'(t)| &= \frac{2}{(2t^2+1)^2}\sqrt{(1-2t^2)^2+(2t)^2+(-2t)^2} \\
&= \frac{2}{(2t^2+1)^2}\sqrt{1-4t^2+4t^4+4t^2+4t^2} \\
&= \frac{2}{(2t^2+1)^2}\sqrt{4t^4+4t^2+1} \\
&= \frac{2}{(2t^2+1)^2}\sqrt{(2t^2+1)^2} \\
&= \frac{2}{(2t^2+1)^2}(2t^2+1) \\
&= \frac{2}{2t^2+1}
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
\mathbf{N}(t) &= \frac{2\langle 1-2t^2, 2t, -2t\rangle}{(2t^2+1)^2}\cdot\frac{2t^2+1}{2} \\
&= \frac{\langle 1-2t^2, 2t, -2t\rangle}{2t^2+1}
\end{aligned}
$$

Then we have $\mathbf{N}(1) = \frac{\langle -1,2,-2\rangle}{3} = \left\langle \frac{-1}{3}, \frac{2}{3}, \frac{-2}{3}\right\rangle$.

Next, we seek the binormal vector $\mathbf{B}(1)$, the vector perpendicular to both the unit tangent and unit normal vector at $t = 1$, i.e. the cross product:

$$
\begin{aligned}
\mathbf{B}(1) = \mathbf{T}(1)\times\mathbf{N}(1) &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{2}{3} & \frac{2}{3} & \frac{1}{3} \\ \frac{-1}{3} & \frac{2}{3} & \frac{-2}{3} \end{vmatrix} \\
&= \mathbf{i}\left(\frac{-4}{9}-\frac{2}{9}\right) - \mathbf{j}\left(\frac{-4}{9}+\frac{1}{9}\right) + \mathbf{k}\left(\frac{4}{9}+\frac{2}{9}\right) \\
&= \left\langle \frac{-2}{3}, \frac{1}{3}, \frac{2}{3}\right\rangle
\end{aligned}
$$

Next, we find the curvature $\kappa(1)$, which measures how quickly the curve $\mathbf{r}(t)$ is changing direction at $t = 1$, which we calculate as follows:

$$
\begin{aligned}
\kappa(t) &= \frac{|\mathbf{T}'(t)|}{|\mathbf{r}'(t)|} = \frac{\frac{2}{2t^2+1}}{|\langle 2t, 2t^2, 1\rangle|} \\
\kappa(1) &= \frac{\frac{2}{3}}{|\langle 2,2,1\rangle|} \\
&= \frac{2}{3}\cdot\frac{1}{\sqrt{4+4+1}} = \frac{2}{9}
\end{aligned}
$$

Finally, we find the arc length of the curve from $t = 0$ to $t = 1$, which is just the length of the curve between these t values, which we find as follows:

$$L = \int_0^1 |\mathbf{r}'(t)| dt = \int_0^1 (2t^2 + 1) dt = \frac{2}{3}t^3 + t\Big|_0^1$$
$$= \frac{2}{3} + 1 - 0 - 0 = \frac{5}{3}$$