1 Highlights from §15.1-15.3

- Disclaimer: This is NOT a complete list of what you need to understand. Any material in the sections may appear on tests.
- We extended differentiation into multiple dimensions in the previous chapter, and now we will do the same for integration.
- In 2D, integrals represented areas under curves. In 3D, they represent volumes under functions of 2 variables.
  - More specifically, if $f(x, y) \geq 0$, then the volume of the solid that lies above rectangle R and below the surface $z = f(x, y)$ is

$$V = \iint\limits_R f(x, y)\mathrm{d}A \tag{1}$$

  - If $R = [a, b] \times [c, d] = \{(x, y) : a \leq x \leq b, c \leq y \leq d\}$, then

$$\iint\limits_R f(x, y)dA = \int_a^b \left[\int_c^d f(x, y)dy\right] dx = \int_a^b \int_c^d f(x, y)dydx \tag{2}$$

    where the integral $\int_c^d f(x, y)dy$ is an integral with respect to y, and x is treated as a constant, which we can calculate.

- §15.3 adapts these ideas to find integrals over general plane regions in the xy-plane as opposed to merely rectangular regions.
  - A plane region D is of **type I** if it lies between the graphs of two continuous functions of x:

$$D = \{(x, y)|a \leq x \leq b, g_1(x) \leq y \leq g_2(x)\}$$

    In this case,

$$\iint\limits_D f(x, y)dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x, y)dydx \tag{3}$$

  - A plane region D is of **type II** if it lies between the graphs of two continuous functions of y:

$$D = \{(x, y)|c \leq y \leq d, h_1(y) \leq x \leq h_2(y)\}$$

    In this case,

$$\iint\limits_D f(x, y)dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x, y)dxdy \tag{4}$$

- Some regions are neither type I nor type II, but can be partitioned into a two or more regions of this type and add the integrals. For example, if $D = D_1 \cup D_2$ where D_1 and D_2 are non-overlapping (or only overlap on the boundaries) regions of type I or II, then

$$\iint_D f(x,y)dA = \iint_{D_1} f(x,y)dA + \iint_{D_2} f(x,y)dA \tag{5}$$

2 Useful Results

– Fubini's Theorem: If f is continuous on the rectangle $R = [a,b] \times [c,d]$, then

$$\iint_R f(x,y)dA = \int_a^b \int_c^d f(x,y)dydx = \int_c^d \int_a^b f(x,y)dxdy \tag{6}$$

That is, for continuous f, we can interchange the order of integration. (The text provides some conditions weaker than continuity which also allow the interchange.)

3 Problems

Example 1 (§15.2) Calculate the integral $\iint_R xye^{x^2y}dA$ where $R = [0,1] \times [0,2]$.

Since xye^{x^2y} is continuous, we can write the multiple integral as iterated integrals by Fubini's Theorem and solve them individually.

We could make the inner integral in terms of x or y, but it seems most conveneint to do the x integral first since it is clear how to calculate it.

$$\begin{aligned}\iint_R xye^{x^2y}dA = \int_0^2 \int_0^1 xye^{x^2y}dxdy &= \int_0^2 \left[\frac{1}{2}e^{x^2y}\right]_{x=0}^{x=1} \\ &= \frac{1}{2}\int_0^2 (e^y - 1)dy = \frac{1}{2}\left[e^y - y\right]_{y=0}^{y=2} \\ &= \frac{1}{2}[(e^2 - 2) - (1 - 0)] = \frac{1}{2}(e^2 - 3)\end{aligned}$$

Example 2 (§15.3) Evaluate $\int_0^{\pi/2} \int_0^{\cos\theta} e^{\sin\theta} dr d\theta$

Since $e^{\sin\theta}$ is just a constant with respect to r, the first integral is very simple

$$\begin{aligned}\int_0^{\pi/2}\int_0^{\cos\theta} e^{\sin\theta} dr d\theta &= \int_0^{\pi/2} \left[re^{\sin\theta}\right]_{r=0}^{r=\cos\theta} d\theta \\ &= \int_0^{\pi/2} \cos\theta e^{\sin\theta} d\theta \\ &= \int_{\sin 0}^{\sin(\pi/2)} e^u du \\ &= [e^u]_{u=0}^{u=1} = e-1\end{aligned}$$

where we used the u substitution $u = \sin\theta$ and $du = \cos\theta d\theta$ to evaluate the second integral.

Example 3 (§15.3, #31) Find the volume of the solid bounded by the cylinder $x^2 + y^2 = 1$ and the planes $z = y$, $x = 0$, $z = 0$ in the first octant.

First, we should notice $x^2 + y^2 = 1$ creates a circle of radius 1 centered at $(0, 0)$ in the xy-plane, and since we consider only the first octant, the region of interest is a quarter circle.

If we consider it as a type I region, we need two functions of x bounding the region. The lower function can be merely $x = 0$ since this plane bounds our region. The upper function comes from $x^2 + y^2 = 1$, which by isolating y is $y = \sqrt{1-x^2}$. We can also take $0 \le y \le 1$.

The height of the region is y since we consider the region between $z = 0$ and $z = y$. With all of this, we can begin calculating the integral

$$\begin{aligned}V &= \int_0^1\int_0^{\sqrt{1-x^2}} y dy dx = \int_0^1 \left[\frac{y^2}{2}\right]_0^{\sqrt{1-x^2}} dx \\ &= \frac{1}{2}\int_0^1 (1-x^2)dx = \frac{1}{2}\left[x - \frac{x^3}{3}\right]_0^1 = \frac{1}{2}\left[1-\frac{1}{3}\right] = \frac{1}{3}\end{aligned}$$

Example 4 (§15.3, #59) Find the average value of $f(x, y) = xy$ over the triangle in the xy-plane bounded by $(0, 0)$, $(1, 0)$, and $(1, 3)$.

The average of a function over a region is the integral over that region divided by the area of that region. The area of the triangle is $\frac{1}{2}bh = \frac{1}{2}(1)(3) = \frac{3}{2}$.

We could do the integral in either order we choose, so let us arbitrarily treat the region as a type I region with $0 \le y \le 3x$, since $y = 3x$ is the line joining

$(0, 0)$ and $(1, 3)$, then we have $0 \leq x \leq 1$. Then

$$\begin{aligned} f_{\text{ave}} &= \frac{1}{3/2} \int_0^1 \int_0^{3x} xydydx = \frac{2}{3} \int_0^1 x \int_0^{3x} ydydx \\ &= \frac{2}{3} \int_0^1 x \left[\frac{y^2}{2} \right]_0^{3x} dx = \frac{2}{3} \int_0^1 \frac{9}{2} x^3 dx = \frac{3}{4} [x^4]_0^1 = \frac{3}{4} \end{aligned}$$